Upper and Lower Bounds in Exponential Tauberian Theorems

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Abstract

In this text we study, for positive random variables, the relation between the behaviour of the Laplace transform near infinity and the distribution near zero. A result of De Bruijn shows that $E(\mathrm{e}^{-\lambda X}) \sim \exp(r\lambda^\alpha)$ for $\lambda \to \infty$ and $P(X \leq \varepsilon) \sim \exp(s/\varepsilon^\beta)$ for $\varepsilon \downarrow 0$ are in some sense equivalent (for $1/\alpha = 1/\beta + 1$) and gives a relation between the constants r and s. We illustrate how this result can be used to obtain simple large deviation results. For use in more complex situations we also give a generalisation of De Bruijn's result to the case when the upper and lower limits are different from each other.

1 Introduction

Tauberian theorems (see for example the monograph Korevaar, 2004) describe the connection between the behaviour of a positive random variable near zero and the behaviour of its Laplace transform near infinity. From De Bruijn's Tauberian theorem (theorem 4.12.9 in Bingham et al., 1987) we can easily conclude the following result.

Theorem 1 *Let $X \geq 0$ be a random variable on a probability space $(\Omega, \mathcal{A}, P)$, $A \in \mathcal{A}$ an event with $P(A) > 0$ and $\alpha \in (0,1)$, $\beta > 0$ with $\frac{1}{\alpha} = \frac{1}{\beta} + 1$. Then the limit*

$$r = \lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \log E(\mathrm{e}^{-\lambda X} \cdot 1_A) \leq 0 \tag{1}$$

exists if and only if

$$s = \lim_{\varepsilon\to 0} \varepsilon^\beta \log P(X \leq \varepsilon, A) \leq 0 \tag{2}$$

exists and in this case we have $|\alpha r|^{1/\alpha} = |\beta s|^{1/\beta}$.

Proof. In theorem 4.12.9 of Bingham et al. (1987) choose their alpha to be $-1/\beta$, $\phi(x) = x^{-1/\beta}$, $\psi(x) = x^{-1/\alpha}$, and $B = |s|$. This gives the proof in the case $A = \Omega$. The case of general sets A can be reduced to the first case by considering the distribution $Q(\,\cdot\,) = P(\,\cdot \cap A)/P(A)$ instead of P. ∎

With the help of this theorem we can use knowledge about the Laplace transform of a given random variable X to show that the probability $P(X \leq \varepsilon)$ for $\varepsilon \downarrow 0$ decays exponentially fast. Therefore in some situations Tauberian theorems of exponential type can be valuable tools for deriving large deviation principles. Typically, in this case one has $\alpha = 1/2$, $\beta = 1$ and thus $s = -r^2/4$. Section 2 illustrates this idea by using theorem 1 to derive a simple large deviation result for the conditional distribution of a Brownian motion, given that the L^2-norm of the path is small.

In general, the limit (2) does not necessarily exist. For large deviation results one usually considers upper and lower limits, and thus theorem 1 cannot be used directly. In section 3 of this text we will therefore derive a version of theorem 1 which considers upper and lower limits. A

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(lengthy) application where upper and lower limits are needed, and where theorem 1 is therefore not enough, can be found in Voss (2008).

The special case of $\alpha = 1/2$ and $\beta = 1$ for the result presented in this text was originally derived as part of my PhD-thesis (Voss, 2004).

2 Brownian Paths with Small L^2-Norm

In this section we illustrate how theorem 1 can be used to derive a simple large deviations principle (LDP) for Brownian motion. See Dembo and Zeitouni (1998) for details about large deviations, and in particular section 5.2 there for large deviation results for Brownian motion. A review of the connections between Tauberian theorems and large deviations, and further references, can be found in Bingham (2008).

Let $\mathcal{X}$ be the space of all paths $\omega\colon [0,t] \to \mathbb{R}$ such that $\omega_0 = 0$, equipped with the topology of pointwise convergence. On $\mathcal{X}$, define a family $(P_\varepsilon)_{\varepsilon>0}$ of measures by

$$P_\varepsilon(A) = \mathbb{W}\Big(A \Bigm| \int_0^t B_s^2\,ds \leq \varepsilon\Big)$$

for all measurable $A \subseteq \mathcal{X}$, where $\mathbb{W}$ is the Wiener measure on $\mathcal{X}$ and B is the canonical process.

Theorem 2 *On the space $\mathcal{X}$ the family $(P_\varepsilon)_{\varepsilon>0}$ satisfies the LDP with the good rate function*

$$I(\omega) = \sup\big\{(t + 2\omega_{t_1}^2 + \cdots + 2\omega_{t_n}^2 + \omega_t^2)^2/8 - t^2/8 \bigm| n \in \mathbb{N}, 0 < t_1 < \cdots < t_n < t\big\}$$

for all $\omega \in \mathcal{X}$.

Proof. Define $X = \int_0^t B_s^2\,ds$. In order to apply theorem 1 we have to calculate the tails of the Laplace transform of X. Formula (1–1.9.7) from Borodin and Salminen (1996) states

$$E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big); B_t \in dz\Big) = \varphi(x;t,z)\,dz$$

where

$$\varphi(x;t,z) = \frac{\sqrt{\gamma}}{\sqrt{2\pi\sinh(t\gamma)}}\exp\Big(-\frac{(x^2+z^2)\gamma\cosh(t\gamma) - 2xz\gamma}{2\sinh(t\gamma)}\Big).$$

For starting point x, measurable sets $A_1, \ldots, A_n \subseteq \mathbb{R}$ and fixed times $0 < t_1 < \cdots < t_n = t$, the Markov property of Brownian motion gives then

$$\begin{aligned} &E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big)1_{A_1}(B_{t_1})\cdots 1_{A_n}(B_{t_n})\Big) \\ &\quad= \int_{A_1}\cdots\int_{A_n} \varphi(x;t_1,z_1)\varphi(z_1;t_2-t_1,z_2) \\ &\qquad\qquad \cdots\varphi(z_{n-1};t_n-t_{n-1},z_n)\,dz_n\cdots dz_1. \end{aligned}$$

We are interested in the exponential tails of this expression for $\gamma \to \infty$.

Let $\varepsilon > 0$. Observe that there are constants $0 < c_1 < c_2$ and $G > 0$ with

$$c_1 e^{-\gamma t/2} \leq \frac{1}{\sqrt{2\pi\sinh(\gamma t)}} \leq c_2 e^{-\gamma t/2} \quad \text{for all } \gamma > G.$$

Furthermore we can use the relation $|2xz| \leq x^2 + z^2$ to get

$$\frac{x^2+z^2}{2}\cdot\frac{\cosh(\gamma t) - 1}{\sinh(\gamma t)} \leq \frac{(x^2+z^2)\cosh(\gamma t) - 2xz}{2\sinh(\gamma t)} \leq \frac{x^2+z^2}{2}\cdot\frac{\cosh(\gamma t)+1}{\sinh(\gamma t)}$$

for all $x, z \in \mathbb{R}$. Because of

$$\frac{\cosh(\gamma t) \pm 1}{\sinh(\gamma t)} = \frac{e^{\gamma t} + e^{-\gamma t} \pm 1}{e^{\gamma t} - e^{-\gamma t}} \longrightarrow 1 \quad \text{for } \gamma \to \infty.$$

we can then find a $\gamma_0 > 0$, such that whenever $\gamma > \gamma_0$ the estimate

$$\frac{x^2+z^2}{2} \cdot (1-\varepsilon) \le \frac{(x^2+z^2)\cosh(\gamma t) - 2xz}{2\sinh(\gamma t)} \le \frac{x^2+z^2}{2} \cdot (1+\varepsilon)$$

holds for all $x, z \in \mathbb{R}$.

Using this estimate we conclude

$$\begin{aligned}
&\limsup_{\gamma\to\infty} \frac{1}{\gamma} \log E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big) 1_{A_1}(B_{t_1})\cdots 1_{A_n}(B_{t_n})\Big) \\
&\quad \le \lim_{\gamma\to\infty} \frac{1}{\gamma} \log \gamma^{n/2} c_2^n \int_{A_1}\cdots\int_{A_n} \mathrm{e}^{-\gamma t_1/2} \exp\Big(-\gamma\frac{x^2+z_1^2}{2}(1-\varepsilon)\Big) \\
&\qquad\qquad \cdot \mathrm{e}^{-\gamma(t_2-t_1)/2} \exp\Big(-\gamma\frac{z_1^2+z_2^2}{2}(1-\varepsilon)\Big) \cdot \cdots \\
&\qquad\qquad \cdot \mathrm{e}^{-\gamma(t_n-t_{n-1})/2} \exp\Big(-\gamma\frac{z_{n-1}^2+z_n^2}{2}(1-\varepsilon)\Big)\, dz_n\cdots dz_1 \\
&\quad = \lim_{\gamma\to\infty} \frac{1}{\gamma} \log \int_{A_1}\cdots\int_{A_n} \exp\big(-\gamma t_n/2 - \gamma(x^2/2 + z_1^2 + \cdots \\
&\qquad\qquad \cdots + z_{n-1}^2 + z_n^2/2)(1-\varepsilon)\big)\, dz_n\cdots dz_1.
\end{aligned}$$

Note the special role of the final point z_n. With the help of the Laplace principle (see *e.g.* Dembo and Zeitouni, 1998, section 4.3) we can calculate the limit on the right hand side to get

$$\begin{aligned}
&\limsup_{\gamma\to\infty} \frac{1}{\gamma} \log E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big) 1_{A_1}(B_{t_1})\cdots 1_{A_n}(B_{t_n})\Big) \\
&\quad \le -\operatorname*{ess\,inf}_{z_1\in A_1,\ldots,z_n\in A_n} \Big(t/2 + (x^2/2 + z_1^2 + \cdots + z_{n-1}^2 + z_n^2/2)(1-\varepsilon)\Big).
\end{aligned}$$

for all $\varepsilon > 0$ and thus

$$\begin{aligned}
&\limsup_{\gamma\to\infty} \frac{1}{\gamma} \log E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big) 1_{A_1}(B_{t_1})\cdots 1_{A_n}(B_{t_n})\Big) \\
&\quad \le -\operatorname*{ess\,inf}_{z_1\in A_1,\ldots,z_n\in A_n} (t/2 + x^2/2 + z_1^2 + \cdots + z_{n-1}^2 + z_n^2/2).
\end{aligned}$$

A very similar calculation gives

$$\begin{aligned}
&\liminf_{\gamma\to\infty} \frac{1}{\gamma} \log E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big) 1_{A_1}(B_{t_1})\cdots 1_{A_n}(B_{t_n})\Big) \\
&\quad \ge -\operatorname*{ess\,inf}_{z_1\in A_1,\ldots,z_n\in A_n} (t/2 + x^2/2 + z_1^2 + \cdots + z_{n-1}^2 + z_n^2/2).
\end{aligned}$$

and together this shows

$$\begin{aligned}
&\lim_{\gamma\to\infty} \frac{1}{\gamma} \log E_x\Big(\exp\big(-\frac{\gamma^2}{2}\int_0^t B_s^2\,ds\big) 1_{A_1}(B_{t_1})\cdots 1_{A_n}(B_{t_n})\Big) \\
&\quad = -\operatorname*{ess\,inf}_{z_1\in A_1,\ldots,z_n\in A_n} (t/2 + x^2/2 + z_1^2 + \cdots + z_{n-1}^2 + z_n^2/2).
\end{aligned} \tag{3}$$

For measurable sets $A_1, \ldots, A_n \subseteq \mathbb{R}$ and fixed times $0 < t_1 < \cdots < t_n = t$, the Tauberian theorem 1 applied to equation (3) gives

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \varepsilon \cdot \log P\Big((B_{t_1}, B_{t_2}, \ldots, B_{t_n}) \in A_1\times A_2\times\cdots\times A_n \;\Big|\; \int_0^t B_s^2\,ds \le \varepsilon\Big) \\
&\quad = \lim_{\varepsilon\downarrow 0} \varepsilon \cdot \log P\Big(B_{t_1}\in A_1, B_{t_2}\in A_2, \ldots, B_{t_n}\in A_n, \int_0^t B_s^2\,ds \le \varepsilon\Big) \\
&\qquad\qquad - \lim_{\varepsilon\downarrow 0} \varepsilon \cdot \log P\Big(\int_0^t B_s^2\,ds \le \varepsilon\Big) \\
&\quad = -\big(t + \operatorname*{ess\,inf}_{z\in A_1\times A_2\times\cdots\times A_n} (2z_1^2 + \cdots + 2z_{n-1}^2 + z_n^2)\big)^2/8 + t^2/8.
\end{aligned} \tag{4}$$

Using $A_n = \mathbb{R}$ we can drop the assumption $t_n = t$ and arrive at the following result. For all measurable sets $A_1, \ldots, A_n \subseteq \mathbb{R}$ and fixed times $0 < t_1 < \cdots < t_n \leq t$ we have

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \varepsilon \cdot \log P\Big((B_{t_1}, B_{t_2}, \ldots, B_{t_n}) \in A_1 \times A_2 \times \cdots \times A_n \Bigm| \int_0^t B_s^2 \, ds \leq \varepsilon \Big) \\ &\qquad = - \operatorname*{ess\,inf}_{z \in A_1 \times A_2 \times \cdots \times A_n} I_{t_1, \ldots, t_n}(z) \end{aligned} \tag{5}$$

where $I_{t_1,\ldots,t_n} \colon \mathbb{R}^n \to \mathbb{R}_+$ is defined by

$$I_{t_1,\ldots,t_n}(z) = \frac{1}{8} \begin{cases} \big(t + 2z_1^2 + \cdots + 2z_n^2\big)^2 - t^2, & \text{if } t_n < t, \text{ and} \\ \big(t + 2z_1^2 + \cdots + 2z_{n-1}^2 + z_n^2\big)^2 - t^2 & \text{for } t_n = t. \end{cases}$$

Since the rate function $I_{t_1,\ldots,t_n}$ is continuous, we can replace ess inf with inf when the sets A_i are open and thus (5) gives an LDP on $\mathbb{R}^n$. From this we can get the LDP on the path space $\mathcal{X}$ with rate function I by applying the Dawson-Gärtner theorem about large deviations for projective limits (see for example Dembo and Zeitouni, 1998, theorem 4.6.1). ∎

Note that the rate function I in the theorem will typically take its infimum for a non-continuous path ω: Assume ω is continuous and non-zero. Let $\varepsilon = \|\omega\|_\infty / 2$. Then we find infinitely many distinct times t with $\omega_t^2 > \varepsilon^2$ and thus $I(\omega) = +\infty$. Therefore it will not be possible to prove the same theorem with $\mathcal{X}$ replaced by $C\big([0,t], \mathbb{R}\big)$.

3 Upper and Lower Limits

In this section we derive an analogue of theorem 1 which considers upper and lower limits. The proof does not rely on theorem 1 and uses only elementary methods.

Theorem 3 *Let $X \geq 0$ be a random variable on a probability space $(\Omega, \mathcal{A}, P)$, $A \in \mathcal{A}$ an event with $P(A) > 0$ and $\alpha \in (0,1)$, $\beta > 0$ with $\frac{1}{\alpha} = \frac{1}{\beta} + 1$.*

a) The upper limits

$$\bar{r} = \limsup_{\lambda \to \infty} \frac{1}{\lambda^\alpha} \log E(\mathrm{e}^{-\lambda X} \cdot 1_A) \quad \textit{and} \quad \bar{s} = \limsup_{\varepsilon \to 0} \varepsilon^\beta \log P(X \leq \varepsilon, A)$$

satisfy $|\alpha \bar{r}|^{1/\alpha} = |\beta \bar{s}|^{1/\beta}$.

b) The lower limits

$$\underline{r} = \liminf_{\lambda \to \infty} \frac{1}{\lambda^\alpha} \log E(\mathrm{e}^{-\lambda X} \cdot 1_A) \quad \textit{and} \quad \underline{s} = \liminf_{\varepsilon \to 0} \varepsilon^\beta \log P(X \leq \varepsilon, A).$$

satisfy $|\alpha \underline{r}|^{1/\alpha} \leq |\beta \underline{s}|^{1/\beta} \leq |\mathrm{e}^{H(\alpha)} \alpha \underline{r}|^{1/\alpha}$ where $H(\alpha) = -\alpha \log(\alpha) - (1-\alpha)\log(1-\alpha)$ and both bounds are sharp.

Note that, because X is positive, the expectation $E(\mathrm{e}^{-\lambda X})$ exists for all $\lambda \geq 0$ and is a number between 0 and 1. Thus the values $\bar{r}$, $\underline{r}$, $\bar{s}$, and $\underline{s}$ will all be negative. Also it is easy to see that theorem 3 does not directly imply theorem 1: If the limit s from theorem 1 exists, then we get

$$|\beta s|^{1/\beta} = |\alpha \bar{r}|^{1/\alpha} \leq |\alpha \underline{r}|^{1/\alpha} \leq |\beta s|^{1/\beta}$$

i.e. the limit r also exists and satisfies $|\alpha r|^{1/\alpha} = |\beta s|^{1/\beta}$. But if we assume that r exists, then theorem 3 only gives

$$|\alpha r|^{1/\alpha} = |\beta \bar{s}|^{1/\beta} \leq |\beta \underline{s}|^{1/\beta} \leq |\mathrm{e}^{H(\alpha)} \alpha r|^{1/\alpha}$$

and we cannot directly conclude that the limit s from theorem 1 exists.

Proof. As in the proof of theorem 1, it is enough to consider the case $A = \mathbb{R}$. Throughout the proof we will use the relations $\beta/\alpha = \beta + 1$ and $\alpha/\beta = 1 - \alpha$ without further comment.

a) The estimate $|\beta \bar{s}|^{1/\beta} \geq |\alpha \bar{r}|^{1/\alpha}$ follows from the exponential Markov inequality: Let $\varepsilon > 0$. From

$$E(\mathrm{e}^{-\lambda X}) \geq \mathrm{e}^{-\lambda \varepsilon} P\big(\mathrm{e}^{-\lambda X} \geq \mathrm{e}^{-\lambda \varepsilon}\big) = \mathrm{e}^{-\lambda \varepsilon} P\big(X \leq \varepsilon\big)$$

we get $P(X \le \varepsilon) \le \mathrm{e}^{\lambda\varepsilon} E(\mathrm{e}^{-\lambda X})$ and thus

$$\varepsilon^\beta \log P(X \le \varepsilon) \le \varepsilon^\beta \big(\lambda\varepsilon + \log E(\mathrm{e}^{-\lambda X})\big) \quad \text{for all } \lambda \ge 0.$$

For $\lambda = (-\frac{\beta}{\beta+1}\bar{r})^{\beta+1}\varepsilon^{-(\beta+1)}$ the bound becomes

$$\varepsilon^\beta \log P(X \le \varepsilon) \le \big(-\frac{\beta}{\beta+1}\bar{r}\big)^{\beta+1} + \big(-\frac{\beta}{\beta+1}\bar{r}\big)^{\beta} \frac{1}{\lambda^\alpha} \log E(\mathrm{e}^{-\lambda X}).$$

Taking upper limits we get

$$\begin{aligned}
\bar{s} &= \limsup_{\varepsilon\downarrow 0} \varepsilon \cdot \log P(X \le \varepsilon) \\
&\le \big(-\frac{\beta}{\beta+1}\bar{r}\big)^{\beta+1} + \big(-\frac{\beta}{\beta+1}\bar{r}\big)^{\beta}\bar{r} = -\frac{\beta^\beta}{(\beta+1)^{\beta+1}}|\bar{r}|^{\beta+1}
\end{aligned}$$

and the claim follows by solving this inequality for $|\beta\bar{s}|^{1/\beta}$.

A more careful analysis is necessary to prove $|\beta\bar{s}|^{1/\beta} \le |\alpha\bar{r}|^{1/\alpha}$. We can express $\bar{r}$ via the lower tails of X:

$$\begin{aligned}
\bar{r} &= \limsup_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \log E(\mathrm{e}^{-\lambda X}) \\
&= \limsup_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \log \int_0^1 P(\mathrm{e}^{-\lambda X} \ge t)\, dt \\
&= \limsup_{\varepsilon\downarrow 0} \varepsilon \log \int_0^\infty P(X \le \varepsilon^{1/\alpha} u)\mathrm{e}^{-u}\, du.
\end{aligned}$$

The definition of $\bar{s}$ gives that for every $\delta > 0$ there exists an $E > 0$, such that for every $\eta < E$ we have $P(X \le \eta) \le \exp\big((\bar{s}+\delta)/\eta^\beta\big)$. Using this estimate and the substitution $v = \varepsilon u$ we find

$$\bar{r} \le \limsup_{\varepsilon\downarrow 0} \varepsilon \log \int_0^\infty \exp\big(-\frac{(|\bar{s}|-\delta)v^{-\beta} + v}{\varepsilon}\big)\, \frac{1}{\varepsilon} dv.$$

The right-hand side can be evaluated by the Laplace principle again and so we find

$$\bar{r} \le -\operatorname*{ess\,inf}_{v\ge 0}\big((|\bar{s}|-\delta)v^{-\beta} + v\big) = -\big(|\bar{s}|-\delta\big)^{1/(\beta+1)}\beta^{-\beta/(\beta+1)}(1+\beta)$$

for every $0 < \delta < |\bar{s}|$ and thus

$$|\bar{r}| \ge |\bar{s}|^{1/(\beta+1)}\beta^{-\beta/(\beta+1)}(1+\beta) = |\bar{s}|^{\alpha/\beta}\frac{\beta^{\alpha/\beta}}{\alpha}.$$

This completes the proof of the bound $|\alpha\bar{r}|^{1/\alpha} \ge |\beta\bar{s}|^{1/\beta}$.

b) Replacing all upper limits with lower limits in the proof of $|\beta\bar{s}|^{1/\beta} \ge |\alpha\bar{r}|^{1/\alpha}$ gives the corresponding bound $|\beta\underline{s}|^{1/\beta} \ge |\alpha\underline{r}|^{1/\alpha}$.

Finally, we prove $|\beta\underline{s}|^{1/\beta} \le |\mathrm{e}^{H(\alpha)}\alpha\underline{r}|^{1/\alpha}$, or equivalently $\underline{r} \le -|\underline{s}|^{1-\alpha}$: Using the estimate $\mathrm{e}^{-\lambda x} \le 1_{[0,\varepsilon]}(x) + \mathrm{e}^{-\lambda\varepsilon}1_{(\varepsilon,\infty)}(x)$ for all $x \ge 0$ gives $E\big(\mathrm{e}^{-\lambda X}\big) \le P(X \le \varepsilon) + \mathrm{e}^{-\lambda\varepsilon}$. Choosing $\varepsilon = \varepsilon(\lambda)$ such that $1/\lambda^\alpha = |\underline{s}|^{-\alpha}\varepsilon^\beta$, we get

$$\frac{1}{\lambda^\alpha}\log E\big(\mathrm{e}^{-\lambda X}\big) \le |\underline{s}|^{-\alpha}\varepsilon^\beta \log\big(P(X \le \varepsilon) + \mathrm{e}^{-|\underline{s}|\varepsilon^{-\beta}}\big).$$

For the second term in the sum, the limit $\lim_{\varepsilon\downarrow 0}\varepsilon^\beta \log \mathrm{e}^{-|\underline{s}|\varepsilon^{-\beta}} = -|\underline{s}|$ exists and thus we can conclude

$$\begin{aligned}
\underline{r} &\le |\underline{s}|^{-\alpha} \liminf_{\varepsilon\downarrow 0} \varepsilon^\beta \log\Big(P(X \le \varepsilon) + \mathrm{e}^{-|\underline{s}|\varepsilon^{-\beta}}\Big) \\
&= |\underline{s}|^{-\alpha} \max\Big(\liminf_{\varepsilon\downarrow 0} \varepsilon^\beta \log P(X \le \varepsilon)\,,\ \lim_{\varepsilon\downarrow 0} \varepsilon^\beta \log \mathrm{e}^{-|\underline{s}|\varepsilon^{-\beta}}\Big) \\
&= |\underline{s}|^{-\alpha} \max\big(-|\underline{s}|\,,\ -|\underline{s}|\big) = -|\underline{s}|^{1-\alpha}.
\end{aligned}$$

This is the required result.

The lower bound on $|\underline{s}|$ is sharp, because in the case of theorem 1 we have equality there. The fact that the upper bound on $|\underline{s}|$ is sharp is shown by the following example. ∎

Example. This example illustrates that the bound $|\beta\underline{s}|^{1/\beta} \leq |e^{H(\alpha)}\alpha\underline{r}|^{1/\alpha}$ is sharp. Let $s < 0$, α and β as above, and $(\varepsilon_n)_{n\in\mathbb{N}_0}$ be a strictly decreasing sequence with $\varepsilon_0 = \infty$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Then we have

$$\sum_{n=1}^{\infty} \Bigl(e^{-|s|\varepsilon_{n-1}^{-\beta}} - e^{-|s|\varepsilon_n^{-\beta}}\Bigr) = e^{-|s|\varepsilon_0^{-\beta}} - \lim_{n\to\infty} e^{-|s|\varepsilon_n^{-\beta}} = 1 - 0 = 1$$

and we can define a random variable X with values in the set $\{\, \varepsilon_n \mid n \in \mathbb{N} \,\}$ by

$$P(X = \varepsilon_n) = e^{-|s|\varepsilon_{n-1}^{-\beta}} - e^{-|s|\varepsilon_n^{-\beta}}$$

for all $n \in \mathbb{N}$. This random variable has

$$P(X \leq \varepsilon) = \sum_{n=n(\varepsilon)}^{\infty} \Bigl(e^{-|s|\varepsilon_{n-1}^{-\beta}} - e^{-|s|\varepsilon_n^{-\beta}}\Bigr) = e^{-|s|\varepsilon_{n(\varepsilon)-1}^{-\beta}}$$

with $n(\varepsilon) = \min\{\, n \in \mathbb{N} \mid \varepsilon_n \leq \varepsilon \,\}$ and consequently

$$\varepsilon^\beta \log P(X \leq \varepsilon) = -|s| \frac{\varepsilon^\beta}{\varepsilon_{n(\varepsilon)-1}^\beta}.$$

By definition of $n(\varepsilon)$ we have $\varepsilon_{n(\varepsilon)} \leq \varepsilon < \varepsilon_{n(\varepsilon)-1}$. This allows us to calculate the exponential tail rates $\underline{s} = s$ and, because s is negative, $\bar{s} = s \cdot \liminf_{n\to\infty} \varepsilon_n^\beta / \varepsilon_{n-1}^\beta$.

Choosing different sequences (ε_n) leads to different values for $\bar{s}$, $\bar{r}$, and $\underline{r}$. For our example let $q < 1$ and define $\varepsilon_n = q^n$ for all $n \in \mathbb{N}$. Then the above calculation shows $\bar{s} = qs$ and $\underline{s} = s$. Theorem 3 gives $\bar{r} = -|\beta qs|^{\alpha/\beta}/\alpha$ and $\underline{r} \in \bigl[-|\beta s|^{\alpha/\beta}/\alpha, -e^{-H(\alpha)}|\beta s|^{\alpha/\beta}/\alpha\bigr] = \bigl[-e^{H(\alpha)}|s|^{\alpha/\beta}, -|s|^{\alpha/\beta}\bigr]$. In order to show that the upper bound on $|\underline{s}|$ is sharp, we have to show that we can have $\underline{r}$ arbitrarily close to $-|s|^{\alpha/\beta}$.

In the situation of the example we can get better bounds on $\underline{r}$ by an explicit calculation. The Laplace transform of X is given by

$$\begin{aligned} E(e^{-\lambda X}) &= \sum_{n\in\mathbb{N}} e^{-\lambda q^n} \bigl(e^{-|s|q^{-\beta(n-1)}} - e^{-|s|q^{-\beta n}}\bigr) \\ &= \sum_{n\in\mathbb{N}} e^{-\lambda q^n - |s|q^{-\beta(n-1)}} \bigl(1 - e^{-|s|(1-q^\beta)q^{-\beta n}}\bigr). \end{aligned}$$

Since $\exp\bigl(-|s|(1-q^\beta)q^{-\beta n}\bigr) \to 0$ as $n \to \infty$, we have $1/2 < 1 - \exp\bigl(-|s|(1-q^\beta)q^{-\beta n}\bigr) < 1$ for sufficiently large n. Define $n(\lambda)$ by $q^{n(\lambda)} \in \bigl[q(|s|/\lambda)^{\alpha/\beta}, (|s|/\lambda)^{\alpha/\beta}\bigr)$. With $f(x) = \exp(-\lambda x - q^\beta|s|x^{-\beta})$ we have

$$E(e^{-\lambda X}) > \exp\bigl(-\lambda q^{n(\lambda)} - |s|q^{-\beta(n(\lambda)-1)}\bigr)\frac{1}{2} = \frac{1}{2} f(q^{n(\lambda)})$$

for sufficiently large λ. Because the only local extremum of f is a local maximum, we can get a lower bound for f on the interval $\bigl[q(|s|/\lambda)^{\alpha/\beta}, (|s|/\lambda)^{\alpha/\beta}\bigr)$ by just considering the boundary points. This leads to

$$E(e^{-\lambda X}) > \frac{1}{2} \min\bigl(f(q(|s|/\lambda)^{\alpha/\beta}), f((|s|/\lambda)^{\alpha/\beta})\bigr) = \frac{1}{2} \exp\bigl(-(1 + \max(q, q^\beta))\lambda^\alpha |s|^{\alpha/\beta}\bigr)$$

for sufficiently large λ. Taking lower limits we get

$$\underline{r} = \liminf_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \log E(e^{-\lambda X}) \geq -\bigl(1 + \max(q, q^\beta)\bigr)|s|^{\alpha/\beta}.$$

By choosing small values of q, we can force $\underline{r}$ to be arbitrarily close to $-|s|^{\alpha/\beta}$ and thus the bound from the theorem is sharp.

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