Sampling Conditioned Hypoelliptic Diffusions

Martin Hairer * Andrew M. Stuart † Jochen Voss ‡

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Abstract

A series of recent articles introduced a method to construct stochastic partial differential equations (SPDEs) which are invariant with respect to the distribution of a given conditioned diffusion. These works are restricted to the case of elliptic diffusions where the drift has a gradient structure, and the resulting SPDE is of second order parabolic type.

The present article extends this methodology to allow the construction of SPDEs which are invariant with respect to the distribution of a class of hypoelliptic diffusion processes, subject to a bridge conditioning, leading to SPDEs which are of fourth order parabolic type. This allows the treatment of more realistic physical models, for example one can use the resulting SPDE to study transitions between meta-stable states in mechanical systems with friction and noise. In this situation the restriction of the drift being a gradient can also be lifted.

1 Introduction

In previous works, see *e.g.* [SVW04, HSVW05, HSV07] or [HSV09] for a review, we described an SPDE-based method to sample paths from SDEs of the form

$$\dot{x}(t) = f\big(x(t)\big) + \dot{w}(t) \qquad \forall t \in [0,T] \tag{1}$$

where $\dot{w}$ is white noise, conditioned on several different types of events. The method works by introducing an 'algorithmic time' τ and constructing a second order SPDE of the form

$$\partial_\tau x(\tau,t) = \partial_t^2 x(\tau,t) + \mathcal{N}\big(x(\tau,t)\big) + \sqrt{2}\,\partial_\tau w(\tau,t) \qquad \forall (\tau,t) \in \mathbb{R}_+ \times [0,T] \tag{2}$$

which has t as its space variable. Here $\partial_\tau w(\tau,t)$ is space-time white noise. The nonlinearity $\mathcal{N}$ and the boundary conditions of the differential operator ∂_t^2 are constructed such that, in stationarity, the distribution of the random function $t \mapsto x(\tau,t)$ coincides with the required conditioned distribution. (See also [RVE05].) It transpires that the distribution of (1) under the bridge conditions $x(0) = x(T) = 0$ corresponds the choice

$$\mathcal{N}_j(x) = -f_i(x)\partial_j f_i(x) - \frac{1}{2}\partial_{ij}^2 f_i(x)$$

(written using Einstein's summation convention) and use of Dirichlet boundary conditions for ∂_t^2.

Assuming ergodicity of the sampling SPDE, one can now solve the sampling problem by simulating a solution to (2) up to a large time τ and then taking $t \mapsto x(\tau,t)$ as an approximation to a path from the conditioned SDE. The resulting sampling method has many applications, some of which are described in [AHSV07]. The biggest restrictions of this method are that the derivation requires the drift f to have some gradient structure and the diffusion matrix (chosen to be the identity matrix in (1) above) to be invertible.

In this article we consider the different problem of sampling conditioned paths of the second order SDE

$$m\ddot{x}(t) = f\big(x(t)\big) - \dot{x}(t) + \dot{w}(t) \qquad \forall t \in [0,T], \tag{3}$$

*Courant Institute, New York University and Mathematics Institute, University of Warwick. Email: mhairer@cims.nyu.edu

†Mathematics Institute, University of Warwick. Email: A.M.Stuart@warwick.ac.uk

‡Department of Statistics, University of Leeds. Email: J.Voss@leeds.ac.uk

conditioned on $x(0) = x_-$ and $x(T) = x_+$. Equation (3) could, for example, describe the time evolution of a noisy mechanical system with inertia and friction. Rewriting this second order SDE as a system of first order SDEs for x and $\dot{x}$ leads to a drift which is in general not a gradient (even in the case when f itself is one) and, since the noise only acts on $\dot{x}$, one obtains a singular diffusion matrix. Thus, this problem is outside the scope of the previous results. However, it has enough structure so that it still can be treated within a similar framework. Indeed, we derive in this article a fourth order SPDE of the form

$$\partial_\tau x(\tau,t) = \big(\partial_t^2 - m^2\partial_t^4\big)x(\tau,t) + \mathcal{N}(x)(\tau,t) + \sqrt{2}\,\partial_\tau w(\tau,t) \qquad \forall (\tau,t) \in \mathbb{R}_+ \times [0,T] \tag{4}$$

where, again, the boundary conditions and the drift term $\mathcal{N}$ are chosen in such a way that the conditioned distribution of (3) is stationary for (4).

One surprising fact about this result is that it does *not* require f to be a gradient. In our earlier works, even the appropriate definition of solutions for the (formal) second order SPDE derived to sample conditioned paths of (1) in the non-gradient case is not clear (see [HSV07, section 9] or [AHSV07, section 9.2]); the analysis for elliptic equations is thus restricted to the gradient case. In contrast, the greater regularity of solutions to SPDE (4) here, sampling conditioned paths of (3), allows us to obtain existence results for the fourth order SPDEs arising, without any gradient requirements on f.

In the special case where f is a gradient and $f(x_-) = f(x_+) = 0$, the components of the nonlinearity $\mathcal{N}$ can be written as

$$\mathcal{N}_j(x) = -f_i(x)\partial_j f_i(x) - m\,\partial_t x_i\,\partial_t x_k\,\partial^2_{jk} f_i(x) + m\,\partial_t\Big(\partial_t x_i\big(\partial_i f_j(x) + \partial_j f_i(x)\big)\Big)$$

(using Einstein's summation convention again). It is tempting to try to derive (2) by taking the limit $m \downarrow 0$ in (4), in particular since the first terms of the corresponding nonlinearities coincide. It transpires that taking this limit is not entirely trivial: one needs to argue that on one hand, $m\,\partial_t x_i\,\partial_t x_k \to \frac{1}{2}\delta_{ik}$ as $m \downarrow 0$, but that the term $m\partial_t\big(\partial_t x_i(\partial_i f_j + \partial_j f_i)\big)$ becomes negligible in the limit. Nevertheless, this argument can be made exact; see [Hai10] for a rigorous derivation of the required limiting procedure.

One novelty of this article compared to earlier work like [Zab88, HSV07] is that there is no natural Banach space (like the space of continuous functions) on which the nonlinearity is well-defined and on which the linearised equation generates a contraction semigroup. The reason for this is that the linear operator of the equations studied in this article is a fourth-order differential operator. Another technical difficulty stems from the fact that the nonlinearity $\mathcal{N}$ has very weak dissipativity and regularity properties.

While preparing this text, we performed some numerical simulations on the fourth order SPDE presented here. Our aim was to study whether the SPDE could be used as the basis of an infinite dimensional MCMC method. Differently from the situation in earlier articles, these simulations proved prohibitively slow and the resulting method does not seem like a useful approach to sampling. This is mainly due to the fact the convergence time to equilibrium seems to grow like T^4 and thus can get very big for non-trivial problems. In the gradient case, since the system converges to the second-order SPDE as $m \to 0$, one could expect improved convergence rates for small values of m. However, the theory developed in [Hai10] suggests that the relevant lengthscale for the small-m problem is m, suggesting that one would need numerical simulations that resolve significantly smaller scales than that in order to obtain reliable results. Again, this would lead to inefficient numerical methods even in the case of small m. Consequently, we do not include our simulation results in this article.

For a number of articles considering fourth-order (S)PDEs, see for example [BMPW01], [DPD96] and [LCM96]. Alternative methods to construct solutions of SPDEs and to identify their stationary distributions are based on the theory of Dirichlet forms, see *e.g.* [RM92].

The text is structured as follows: in section 2 we give a detailed description of the sampling problem under consideration and formulate the main result in theorem 4. The proof of this result is given in sections 3, 4 and 5.

Notation

Throughout the article, we will use the notation as introduced above: by $s, t \in [0,T]$ we denote 'physical time', *i.e.* the time variable in equations like (1) and (3) which define the target

distributions. By $\sigma, \tau \geq 0$ we denote 'algorithmic time', *i.e.* the time variable in sampling equations like (2) and (4). Thus, in the sampling SPDEs, τ takes the role of time and t takes the role of space.

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2 The Sampling Problem

In this section we give the full statement of the sampling problem we want to solve; the main result is contained in theorem 4.

First consider the following unconditioned second order SDE:

$$\begin{aligned} m\,\ddot{x}(t) = f\big(x(t)\big) - \dot{x}(t) + \dot{w}(t) \qquad \forall t \in [0,T], \\ x(0) = x_0, \quad \dot{x}(0) = v_0 \end{aligned} \tag{5}$$

where the solution x takes values in $\mathbb{R}^d$, $m > 0$ is a constant, $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a given function, and w is a standard Brownian motion on $\mathbb{R}^d$. The initial conditions x_0 and v_0 are either deterministic, or random variables independent of w. The solution to this SDE can be interpreted as the time evolution of the state of a mechanical system with friction under the influence of noise. In this case m would be the *mass* and f would be an *external force field*. Models like this are, for example, widely used in molecular dynamics since for conservative forces f they describe Hamiltonian systems in contact with a heat bath. In this context, equation (5) is called the *Langevin equation*. The limiting case $m = 0$ corresponds to the *Brownian dynamics* (1).

Remark 1. Arbitrary constants in front of the $\dot{x}$ and $\dot{w}$ terms can be introduced using a scaling argument: Let $\beta, \gamma > 0$ and define the process y by $y(t) = x(t/\gamma)/\sqrt{\beta/2}$. Then y solves the SDE

$$\tilde{m}\,\ddot{y}(t) = \tilde{f}\big(y(t)\big) - \gamma\,\dot{y}(t) + \sqrt{\frac{2\gamma}{\beta}}\,\dot{w}(t) \qquad \forall t \in [0,\tilde{T}]$$

where $\tilde{m} = \gamma^2 m$, $\tilde{f}(x) = f\big(\sqrt{\beta/2}x\big)/\sqrt{\beta/2}$ and $\tilde{T} = \gamma T$. Thus, by rescaling T, m and F we can assume $\beta = 2$ and $\gamma = 1$ without loss of generality.

For our analysis we rewrite the second order SDE (5) as a system of first-order SDEs in the variables x and $\dot{x}$. We get

$$\begin{aligned} dx(t) &= \dot{x}(t)\,dt, & x(0) &= x_0, \\ m\,d\dot{x}(t) &= f\big(x(t)\big)\,dt - \dot{x}(t)\,dt + dw(t) & \dot{x}(0) &= v_0. \end{aligned} \tag{6}$$

In the Hamiltonian case $f(x) = -\nabla V(x)$, and provided that the potential V is sufficiently regular, it can be checked that the Boltzmann-Gibbs distribution

$$\exp\Big(-2\big(V(x) + \frac{1}{2}m\dot{x}^2\big)\Big)\,d\dot{x}\,dx$$

is invariant for (6). If V is sufficiently coercive, this distribution can be normalised to a probability distribution. Note that in equilibrium, the position x and the velocity $\dot{x}$ are independent. Thus, in stationarity, the velocity satisfies $\dot{x}(t) \sim \mathcal{N}\big(0, 1/2m\big)$ for all $t \in [0,T]$. We will, even for the non-gradient case, use this distribution for the initial condition for $\dot{x}$.

Definition 2. For $T > 0$, $x_-, x_+ \in \mathbb{R}^d$ and $f\colon \mathbb{R}^d \to \mathbb{R}^d$, define Q_f^{0,x_-} to be the distribution of the process x given by (6) where $x_0 = x_-$ and $v_0 \sim \mathcal{N}\big(0, 1/2m\big)$, independent of w. Define the *target distribution* $Q_f^{0,x_-;T,x_+}$ to be the distribution of x under Q_f^{0,x_-}, conditioned on $x(T) = x_+$.

The sampling problem considered in the rest of this article is to find a stochastic process with values in $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$ which has the target distribution $Q_f^{0,x_-;T,x_+}$ as its stationary distribution. Note that $Q_f^{0,x_-;T,x_+}$ is just the distribution of x and not of the pair $(x,\dot{x})$ and thus is a probability measure on $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$. Considering this distribution is sufficient since for solutions of (6) the initial condition $x(0)=x_-$ allows to find a bijection between the paths x and $\dot{x}$. If f is a gradient, the distribution $Q_f^{0,x_-;T,x_+}$ coincides with the distribution of the process in stationarity, conditioned on $x(0)=x_-$ and $x(T)=x_+$.

Definition 3. Let L denote the formal differential operator

$$L = -m^2\partial_t^4 + \partial_t^2$$

and define $\mathcal{L}$ to be this differential operator on the space $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$, equipped with the domain

$$\mathcal{D}(\mathcal{L}) = \big\{x\in \mathrm{H}^4 \;\big|\; x(0)=x(T)=0,\; m\,\partial_t^2 x(0) = \partial_t x(0),\; m\,\partial_t^2 x(T) = -\partial_t x(T)\big\} \tag{7}$$

where $\mathrm{H}^4 = \mathrm{H}^4\big([0,T],\mathbb{R}^d\big)$ is the Sobolev space of functions with square integrable generalised derivatives up to the fourth order. Furthermore, let $\bar{x}\colon [0,T]\to\mathbb{R}^d$ be the solution of the boundary value problem $L\bar{x}=0$ with boundary conditions

$$\bar{x}(0)=x_-,\quad \bar{x}(T)=x_+,\quad m\,\partial_t^2\bar{x}(0)=\partial_t\bar{x}(0),\quad m\,\partial_t^2\bar{x}(T)=-\partial_t\bar{x}(T). \tag{8}$$

We will see in lemma 17 that the operator $\mathcal{L}$ given by this definition is self-adjoint and negative definite.

Theorem 4. *Consider the* $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$*-valued equation*

$$dx(\tau) = \mathcal{L}(x(\tau)-\bar{x})\,d\tau + \mathcal{N}\big(x(\tau)\big)\,d\tau + \sqrt{2}\,dw(\tau), \qquad x(0)=x_0. \tag{9}$$

Here $\mathcal{L}$ *and* $\bar{x}$ *are given in definition 3,* w *is a cylindrical Wiener process,* $x_0\in\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$ *and*

$$\begin{aligned}\mathcal{N}_k(x) = &-f_i(x)\partial_k f_i(x) \;+\; m\,\partial_t x_i\,\partial_t x_j\,\partial_{ij}^2 f_k(x)\\ &-\partial_t x_i\big(\partial_i f_k(x) - \partial_k f_i(x)\big) \;+\; m\,\partial_t^2 x_i\big(\partial_i f_k(x) + \partial_k f_i(x)\big)\\ &+ m\big(f_k(x_-)\partial_t\delta_0 - f_k(x_+)\partial_t\delta_T\big)\end{aligned} \tag{10}$$

for $k=1,\dots,d$ *where we used Einstein's summation convention over repeated indices,* δ_0 *and* δ_T *are the Dirac distributions at* 0 *and* T *respectively, and all derivatives are taken in the distributional sense. Assume that* $f\in\mathrm{C}^2(\mathbb{R}^d,\mathbb{R}^d)$*, that the partial derivatives* $\partial_i f$ *and* $\partial_{ij}f$ *are bounded and globally Lipschitz continuous for all* $i,j=1,\dots,d$ *and that there are constants* $\beta<1$ *and* $c>0$ *such that* $|f(x)|\le|x|^\beta+c$ *for all* $x\in\mathbb{R}^d$*. Furthermore assume that the SDE* (6) *a.s. has a solution up to time* T*. Then the following statements hold:*

a) For every $x_0\in\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$*, equation* (9) *has a unique, global, continuous mild solution with* $\mathbb{E}\big(\|x(\tau)\|_{\mathrm{L}^2}^2\big)<\infty$ *for all* $\tau>0$*.*

b) For $\tau>0$*, the solution* $x(\tau)$ *a.s. takes values in the Sobolev space* $\mathrm{H}^1\big([0,T],\mathbb{R}^d\big)$*.*

c) The distribution $Q_f^{0,x_-;T,x_+}$ *given by definition 2 is invariant for* (9)*.*

Remark 5. The sub-linear growth-condition $|f(x)|\le|x|^\beta+c$ on the drift seems quite technical. The condition is only required for the bounds in lemma 22. We believe that an additional, linear drift term can be added by incorporating it into the linear operator $\mathcal{L}$, following a similar procedure as in [HSVW05].

Remark 6. As in [HSV07, remark 5.5], we see that the terms involving derivatives of Dirac distributions can be interpreted as modifications to the boundary conditions. Proceeding this way, we see that (9) is formally equivalent to the SPDE

$$\begin{aligned}\partial_\tau x_k(\tau,t) = &\,Lx_k(\tau,t) - f_i(x)\partial_k f_i(x) + m\,\partial_t x_i\,\partial_t x_j\,\partial_{ij}^2 f_k(x)\\ &-\partial_t x_i\big(\partial_i f_k(x)-\partial_k f_i(x)\big) + m\,\partial_t^2 x_j\big(\partial_j f_k(x)+\partial_k f_j(x)\big) + \sqrt{2}\,\partial_\tau w_k(\tau,t),\end{aligned}$$

where $\partial_\tau w$ is space-time white noise, endowed with the boundary conditions

$$\begin{aligned} x(0) &= x_- & x(T) &= x_+ \\ m\,\partial_t^2 x(0) &= \partial_t x(0) + f(x_-), & m\,\partial_t^2 x(T) &= -\partial_t x(T) + f(x_+). \end{aligned}$$

In the one-dimensional case $f\colon \mathbb{R} \to \mathbb{R}$ this SPDE simplifies further to

$$\partial_\tau x(\tau,t) = Lx(\tau,t) - f(x)f'(x) + m\,(\partial_t x)^2\,f''(x) + 2m\,\partial_t^2 x\,f'(x) + \sqrt{2}\,\partial_\tau w(\tau,t).$$

Remark 7. Using a standard bootstrapping argument like the one in the proof of [Hai09, Theorem 6.5] one can show that, in fact, the solution x of (9) takes values in the Sobolev space $\mathrm{H}^r\big([0,T],\mathbb{R}^d\big)$ for every $r < 3/2$.

The remainder of the article gives a proof of theorem 4. We start the argument, in section 3, by collecting some results about the differential operator $\mathcal{L}$. Section 4 shows that the theorem holds for the linear case $f \equiv 0$, in which case $\mathcal{N} \equiv 0$. Finally, Section 5 then completes the proof by showing that introduction of the drift $\mathcal{N}$ changes the stationary distribution of (9) in the correct way to account for a non-vanishing f.

3 Analysis of the Linear Operator

This section collects some results about the operator $\mathcal{L}$ from definition 3. Since we are only interested in the operator itself and not in the full SPDE, in addition to the scaling-argument from remark 1, we can rescale t. Thus, throughout section 3, we will consider the operator $\bar{\mathcal{L}}$ defined as

$$\bar{\mathcal{L}} = -\partial_t^4 + \gamma^2\partial_t^2,$$

where $\gamma = \frac{T}{\pi m}$, on the domain

$$\mathcal{D}(\bar{\mathcal{L}}) = \big\{x \in \mathrm{H}^4 \;\big|\; x(0) = x(\pi) = 0,\ \frac{T}{\pi\gamma}\,\partial_t^2 x(0) = \partial_t x(0),\ \frac{T}{\pi\gamma}\,\partial_t^2 x(\pi) = -\partial_t x(\pi)\big\}.$$

Then, after rescaling t, $\bar{\mathcal{L}}$ differs from the operator $\mathcal{L}$ from definition 3 only by multiplication of a positive constant.

Throughout the rest of the paper we will use the following notation: We denote by $(S(\tau))_{\tau\geq 0}$ the semigroup associated to $\bar{\mathcal{L}}$ on $\mathcal{H} = \mathrm{L}^2\big([0,\pi],\mathbb{R}^d\big)$ and by $\mathcal{H}_\alpha = \mathcal{D}\big((-\bar{\mathcal{L}})^\alpha\big)$ the associated interpolation spaces.

3.1 Approximation to the Spectral Decomposition

Lemma 8. *$\bar{\mathcal{L}}$ is a self-adjoint, negative definite operator on $\mathrm{L}^2\big([0,\pi],\mathbb{R}^d\big)$.*

Proof. Using partial integration it is easy to see that

$$\begin{aligned} \langle x, \mathcal{L}y\rangle = -m^2\int_0^\pi \partial_t^2 x\,\partial_t^2 y\,dt - \int_0^\pi \partial_t x\,\partial_t y\,dt \\ - m\Big(\partial_t x(0)\partial_t y(0) + \partial_t x(\pi)\partial_t y(\pi)\Big) \end{aligned}$$

for all $x, y \in \mathcal{D}(\mathcal{L})$, *i.e.* the operator $\mathcal{L}$ is symmetric and negative. Its self-adjointness can be checked just as in [RS72, section VIII]. ∎

Lemma 9. *Let λ_k, $k \in \mathbb{N}$ be the eigenvalues of $-\bar{\mathcal{L}}$ and e_k be the corresponding eigenfunctions. Define furthermore*

$$\big(f_k^{(i)} \;\big|\; i = 1,2,3,4\big) = \big(\sin kt,\ \cos kt,\ e^{-kt},\ e^{-k(\pi-t)}\big).$$

Then the following statements hold:
a) The eigenvalues of $\bar{\mathcal{L}}$ satisfy $\lambda_k = k^4 + \mathcal{O}(k^2)$.

b) There exist functions $g_k^{(i)}$ *such that* $e_k(t) = \sin(kt) + \frac{1}{k}\sum_{j=1}^4 g_k^{(i)}(t) f_k^{(i)}(t)$ *for all* $t \in [0,\pi]$ *and such that* $\sup_{j=1}^4 \sup_{k\in\mathbb{N}} \|g_k^{(j)}\|_{\mathrm{C}^j} < \infty$ *for every* $j \geq 0$.

Proof. Since $\bar{\mathcal{L}}$ acts independently on each coordinate, we can assume $d = 1$ without loss of generality. The eigenfunctions of $\bar{\mathcal{L}}$ can be written in the form

$$x(t) = \xi_1 \mathrm{e}^{\kappa_+(t-\pi)} + \xi_2 \mathrm{e}^{-\kappa_+ t} + \xi_3 \mathrm{e}^{i\kappa_- t} + \xi_4 \mathrm{e}^{-i\kappa_- t},$$

where

$$\kappa_\pm = \sqrt{\sqrt{\mu^4 + \frac{\gamma^4}{4}} \pm \frac{\gamma^2}{2}} = \mu \pm \frac{\gamma^2}{4\mu} + \mathcal{O}(1/\mu^3),$$

with $\lambda = \mu^4$ the corresponding eigenvalue. The coefficient vector $\xi \in \mathbb{C}^4$ is determined by the boundary conditions: for x to be an eigenfunction of $\bar{\mathcal{L}}$, ξ must satisfy $A_\mu \xi = 0$ where

$$A_\mu = \begin{pmatrix} \mathrm{e}^{-\kappa_+\pi} & 1 & 1 & 1 \\ 1 & \mathrm{e}^{-\kappa_+\pi} & \mathrm{e}^{i\kappa_-\pi} & \mathrm{e}^{-i\kappa_-\pi} \\ (\alpha\kappa_+ - 1)\mathrm{e}^{-\kappa_+\pi} & \alpha\kappa_+ + 1 & -\alpha\kappa_- - i & -\alpha\kappa_- + i \\ \alpha\kappa_+ + 1 & (\alpha\kappa_+ - 1)\mathrm{e}^{-\kappa_+\pi} & (-\alpha\kappa_- + i)\mathrm{e}^{i\kappa_-\pi} & (-\alpha\kappa_- - i)\mathrm{e}^{-i\kappa_-\pi} \end{pmatrix}$$

and $\alpha = T/\pi\gamma$. Setting $\kappa = (\kappa_+ + \kappa_-)/2 = \mu + \mathcal{O}(1/\mu^3)$ for ease of notation, we note that this equation has non-zero solutions if and only if

$$\begin{aligned} 0 = \det A_\mu &= 8i\big(\big(\alpha^2\kappa^2 + \alpha\kappa\big)\sin(\kappa_-\pi) - \big(\tfrac{1}{2} + \alpha\kappa\big)\cos(\kappa_-\pi)\big) + \mathcal{O}(1/\mu) \\ &= 8i\alpha^2\mu^2 \sin\mu\pi + \mathcal{O}(\mu). \end{aligned}$$

It follows immediately that, at least for large values of μ, one has $\mu = k + \mathcal{O}(1/k)$ with $k \in \mathbb{N}$, so that $\lambda_k = k^4 + \mathcal{O}(k^2)$ as requested. In particular, one has

$$\kappa_\pm = k + \frac{\beta_\pm}{k} + \mathcal{O}(1/k^2),$$

for some constants $\beta_\pm \in \mathbb{R}$. It remains to check the statement about the eigenfunctions.

Given that we already have good control on the eigenvalues, our claim will follow if we are able to show that one can choose $\xi = (0, 0, \frac{1}{2}, -\frac{1}{2}) + \mathcal{O}(1/k)$. Expanding A_μ in powers of k, we obtain

$$A_\mu = k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & -\alpha & -\alpha \\ \alpha & 0 & -\alpha & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & (-1)^k & (-1)^k \\ 0 & 1 & -i & i \\ 1 & 0 & ic(-1)^k & -ic(-1)^k \end{pmatrix} + \mathcal{O}(1/k) \equiv kA_\mu^{(0)} + A_\mu^{(1)} + \mathcal{O}(1/k),$$

for $c = 1 - \alpha\beta_-\pi$. It now follows from standard perturbation theory (see *e.g.* [Kat66, theorem II.5.4]) that the eigenvector ξ with eigenvalue 0 can be written as $\xi = \xi^{(0)} + \mathcal{O}(1/k)$, where $\xi^{(0)}$ satisfies $A_\mu^{(0)}\xi^{(0)} = 0$. Since $A_\mu^{(0)}$ is degenerate, this is however not sufficient to determine $\xi^{(0)}$ uniquely, but only tells us that $\xi^{(0)}$ is of the form $(a+b, a+b, a, b)$ for $a, b \in \mathbb{R}$. In order to determine a and b, we have to consider the next order, which yields the compatibility condition $A_\mu^{(1)}\xi^{(0)} \in \mathrm{Range}\, A_\mu^{(0)}$. This compatibility condition can be rewritten as $a + b = 0$, so that we can indeed choose $\xi^{(0)} = (0, 0, \frac{1}{2}, -\frac{1}{2})$, as requested. ∎

3.2 The Relation Between Interpolation and Sobolev Spaces

In this section, we show how the interpolation spaces $\mathcal{H}_\alpha$ associated to the operator $\bar{\mathcal{L}}$ relate to the usual fractional Sobolev spaces. These results are "well-known" in the folklore of the subject. However, in our specific context (especially since we need to consider fractional exponents), we were not able to derive them as straightforward corollaries from results in standard textbooks on function spaces, like [Tri83, Tri92, Tri06]. Because of this, and since one can find rather short and self-contained proofs, we prefer to include them here.

Before we turn to this however, we start with a comparison between the interpolation spaces of the Dirichlet Laplacian and the periodic Laplacian. These are going to be useful in the sequel.

Let Δ_0 denote the Laplacian on $[0,\pi]$ with Dirichlet boundary conditions and let Δ denote the Laplacian on $[0,2\pi]$ with periodic boundary conditions. These operators are self-adjoint in $H_0 = \mathrm{L}^2\big([0,\pi]\big)$ and $H = \mathrm{L}^2\big([0,2\pi]\big)$ respectively. We denote by H_0^s the domain of $\Delta_0^{s/2}$ and by H^s the domain of $\Delta^{s/2}$ (defined in the usual way through spectral decomposition). The aim of this section is to study the correspondence between these two different types of fractional Sobolev spaces. Denote by $\iota\colon \mathrm{H}_0^s \to \mathrm{H}^s$ the map

$$\iota f(t) = \begin{cases} f(t) & \text{for } t \in [0,\pi], \\ -f(2\pi - t) & \text{for } t \in (\pi, 2\pi]. \end{cases}$$

Note that $\iota/2$ is an isometry since it maps the eigenfunctions of Δ_0 into eigenfunctions of Δ. This therefore defines an inclusion $\mathrm{H}_0^s \subseteq \mathrm{H}^s$. A natural left inverse for ι is given by the restriction map

$$rf = f\big|_{[0,\pi]} \qquad \text{for all } f \in \mathrm{H}^s.$$

However, r is not an isometry and, for $s \geq 1/2$, it certainly doesn't map H^s into H_0^s in general (since the constant function 1 belongs to every H^s but only belongs to H_0^s for $s < 1/2$). We do however have the following:

Lemma 10. *The restriction operator r is bounded from H^s into H_0^s for any $s < 1/2$.*

Proof. Note first that H^s is isomorphic to H via the isomorphism $x \mapsto \Delta^{s/2}x$ and similarly for H_0^s, so that the study of r as an operator from H^s to H_0^s is equivalent to the study of the operator $\Delta_0^{s/2} r \Delta^{-s/2}$ from H to H_0. Furthermore, we know that r is bounded from H to H_0, so that it suffices to show that the operator $A = \big(\Delta_0^{s/2} r \Delta^{-s/2} - r\big)$ is bounded from H to H_0.

Since A maps $\sin(n\,\cdot)$ to 0 for every n, it suffices to consider A on the subspace of H given by even functions, and generated by the basis of eigenfunctions of Δ given by $\varphi_n(t) = \frac{1}{\pi}\cos nt$. Define furthermore the basis of eigenfunctions of Δ_0 given by $\psi_m = \frac{2}{\pi}\sin mt$. This yields for A the matrix elements

$$\begin{aligned} A_{mn} = \langle \psi_m, A\varphi_n\rangle &= \frac{2}{\pi^2}(m^s n^{-s} - 1)\int_0^\pi \sin(mt)\cos(nt)\,dt \\ &= \begin{cases} \frac{4m(m^s n^{-s}-1)}{\pi^2(m+n)(m-n)} & \text{if } m+n \text{ is odd}, \\ 0 & \text{if } m+n \text{ is even}. \end{cases} \end{aligned}$$

It follows that there exists a constant $C > 0$ such that $\|A\| \leq C\|\hat{A}\|$, where the operator $\hat{A}$ is defined via its matrix elements by

$$\hat{A}_{mn} = \begin{cases} mn^{-2} & \text{if } n \geq m, \\ m^{s-1}n^{-s} & \text{if } m \geq n. \end{cases}$$

Now it is a straightforward exercise in linear algebra to show that, given an orthonormal basis $\{\varphi_n\}_{n\geq 0}$, an operator $\hat{A}$ is bounded if there exist positive numbers $f_{m,n}$ such that the inequalities

$$\sup_{n\geq 0}\sum_{m\geq 0}\frac{|\langle \hat{A}\varphi_n, \hat{A}\varphi_m\rangle|}{f_{n,m}} < \infty, \qquad \sup_{m\geq 0}\sum_{n\geq 0}|\langle \hat{A}\varphi_n, \hat{A}\varphi_m\rangle| f_{n,m} < \infty, \tag{11}$$

both hold. (Just expand $\|\hat{A}x\|^2$ for $x = \sum_{n\geq 0} x_n\varphi_n$ and make use of the inequality $|x_n x_m| \leq \frac{x_n^2}{2f_{n,m}} + \frac{x_m^2 f_{n,m}}{2}$.) We will show that (11) does indeed hold for $\hat{A}$ as above. Assuming without loss of generality that $m \geq n$, we have the bound

$$\begin{aligned} |\langle \hat{A}\varphi_n, \hat{A}\varphi_m\rangle| &\leq \sum_{k=1}^{n} k^2 m^{-2} n^{-2} + \sum_{k=n}^{m} k^s m^{-2} n^{-s} + \sum_{k=m}^{\infty} k^{2s-2} m^{-s} n^{-s} \\ &\leq C\big(nm^{-2} + m^{s-1}n^{-s}\big) \leq Cm^{s-1}n^{-s}, \end{aligned}$$

and similarly for $n \geq m$. Here, we have made use of the fact that $s < \frac{1}{2}$ to ensure that the last sum converges. It remains to check that the bounds (11) are satisfied for some choice of $f_{m,n}$. With the choice $f_{m,n} = \sqrt{m/n}$, we obtain

$$\sum_{m\geq 0} \frac{|\langle \hat{A}\varphi_n, \hat{A}\varphi_m\rangle|}{f_{n,m}} \leq C \sum_{m=1}^{n} m^{-s-\frac{1}{2}} n^{s-\frac{1}{2}} + C \sum_{m=n}^{\infty} m^{s-\frac{3}{2}} n^{\frac{1}{2}-s} \leq C,$$

where we made again use of the fact that $s < \frac{1}{2}$. The second bound in (11) is obtained in an identical way with the roles of m and n reversed. ∎

For $s > 1/2$, the problem is that elements of H_0^s are forced to be equal to 0 at the boundary, which is not the case for elements of H^s. One has however:

Lemma 11. *For any $s \in (1/2, 2]$, the map r is bounded from the subspace of H^s consisting of functions that vanish at 0 and π into H_0^s.*

Proof. Instead of considering the restriction operator r as before, we are going to consider the operator $\tilde{r}$ defined on continuous functions as

$$\big(\tilde{r}f\big)(t) = f(t) - \frac{1}{\pi}\big(f(0)(\pi - t) + f(1)t\big).$$

Note that $\tilde{r}f = rf$ if $f(0) = f(\pi) = 0$, so that the statement will be implied by the fact that $\tilde{r}$ is shown to be a bounded operator from H^s to H_0^s. Therefore, instead of considering A as before, we consider the operator $\tilde{A} = \big(\Delta_0^{s/2}\tilde{r}\Delta^{-s/2} - r\big)$, which has matrix elements

$$\begin{aligned}\tilde{A}_{mn} &= \begin{cases} A_{mn} - \frac{4}{\pi^2} m^{s-1} n^{-s} & m+n \text{ odd}, \\ 0 & m+n \text{ even}. \end{cases} \\ &= \begin{cases} \frac{4}{\pi^2}\frac{m}{m^2-n^2}\big(m^{s-2}n^{2-s} - 1\big) & m+n \text{ odd}, \\ 0 & m+n \text{ even}. \end{cases}\end{aligned}$$

Note that $s = 2$ is a special case since one then has $\tilde{A} = 0$ as a consequence of the relation $\Delta_0 \tilde{r} = r\Delta$.

In this regime, we have as before $\|\tilde{A}\| \leq C\|\hat{A}\|$, but this time $\hat{A}$ is defined via its matrix elements by

$$\hat{A}_{mn} = \begin{cases} m^{s-1}n^{-s} & \text{if } n \geq m, \\ m^{-1} & \text{if } m \geq n. \end{cases}$$

Computing $\langle \hat{A}\varphi_m, \hat{A}\varphi_n\rangle$ for $m \geq n$ as before, we note that there is a difference between the case $s \leq 1$ and the case $s \geq 1$. We obtain:

$$|\langle \hat{A}\varphi_m, \hat{A}\varphi_n\rangle| \leq \begin{cases} m^{-1} & s \geq 1, \\ n^{s-1}m^{-s} & s < 1. \end{cases}$$

For $s < 1$, we now make the choice $f_{m,n} = m^{s-\varepsilon}n^{\varepsilon-s}$, where $\varepsilon > 0$ is chosen sufficiently small so that $2s - \varepsilon > 1$ (this is always possible since $s > \frac{1}{2}$). With this choice, we obtain

$$\sum_{m\geq 0} \frac{|\langle \hat{A}\varphi_n, \hat{A}\varphi_m\rangle|}{f_{n,m}} \leq C \sum_{m=1}^{n} m^{\varepsilon-1} n^{-\varepsilon} + C \sum_{m=n}^{\infty} m^{\varepsilon-2s} n^{2s-1-\varepsilon} \leq C,$$

and similarly for the other term. This calculation also works for the case $s = 1$, so that the case $s \geq 1$ can be obtained in an identical manner (set for example $\varepsilon = \frac{1}{2}$). ∎

Consider now the operator $\mathcal{L}_a$ given by $\big(\mathcal{L}_a f\big)(t) = \partial_t^4 f$, endowed with the boundary conditions $f(0) = f(\pi) = 0$ and $f''(0) = -af'(0)$, $f''(\pi) = af'(\pi)$. Since the domain of the square of the Dirichlet Laplacian is $\mathcal{D}(\Delta_0^2) = \big\{f \in \mathcal{D}(\Delta_0) \;\big|\; \Delta_0 f \in \mathcal{D}(\Delta_0)\big\} = \mathcal{D}(\mathcal{L}_0)$ we have $\mathcal{L}_0 = \Delta_0^2$. The following lemma shows that $\mathcal{L}_a$ for $a \neq 0$ can still be viewed as a perturbation of Δ_0^2.

Proposition 12. *Fix $a \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary and define the linear operator $A\colon \mathrm{H}_0^{\frac{3}{2}+\varepsilon} \to \mathrm{H}_0^{-\frac{3}{2}-\varepsilon}$ by*

$$Af = f'(0)\delta_0' - f'(\pi)\delta_\pi'.$$

Then, the operator $\tilde{\mathcal{L}}_a = \Delta_0^2 + aA$ is the generator of an analytic semigroup on H_0. Furthermore, this semigroup coincides with the one generated by $\mathcal{L}_a$, so that $\tilde{\mathcal{L}}_a = \mathcal{L}_a$. As a consequence, we obtain the identities $\mathcal{H}^\alpha = \mathrm{H}_0^{4\alpha}$ for every $\alpha \in (-\frac{5}{8}, \frac{5}{8})$.

Proof. Note first that A is well-defined since it follows from standard Sobolev embedding theorems that f' is continuous for every $f \in \mathrm{H}^{\frac{3}{2}+\varepsilon}$ and therefore for every $f \in \mathrm{H}_0^{\frac{3}{2}+\varepsilon}$. Thus, δ_0' and δ_π' can be considered as elements of the Sobolev space $\mathrm{H}_0^{-(\frac{3}{2}+\varepsilon)}$. Since Δ_0^2 generates an analytic semigroup on H_0^α for any $\alpha \in \mathbb{R}$, it follows from applying [Hai09, Proposition 4.42] once with $\mathcal{B} = \mathrm{H}_0^{-\frac{3}{2}}$ and once with $\mathcal{B} = \mathrm{H}_0^{-\frac{5}{2}+\varepsilon}$ that $\tilde{\mathcal{L}}_a$ is the generator of an analytic semigroup on H_0^α for every $\alpha \in (-\frac{5}{2}, \frac{5}{2})$ and that the corresponding scale of interpolation spaces satisfies $\tilde{\mathcal{H}}^\alpha = \mathrm{H}_0^{4\alpha}$ for every $\alpha \in (-\frac{5}{8}, \frac{5}{8})$.

It therefore remains to show that the semigroup $\tilde{S}_\tau$ generated by $\tilde{\mathcal{L}}_a$ coincides with the semigroup S_τ generated by $\mathcal{L}_a$. Since, for any $u \in \mathrm{H}_0$, any $\tau > 0$ and any $t \in (0, \pi)$, we have the identities

$$\partial_\tau \tilde{S}u(\tau, t) = -\partial_t^4 \tilde{S}u(\tau, t), \qquad \partial_\tau Su(\tau, t) = -\partial_t^4 Su(\tau, t),$$

it suffices to show that $\tilde{S}_\tau u \in \mathcal{D}(\mathcal{L}_a)$ for $\tau > 0$. Since we already know that $\tilde{\mathcal{H}}^{\frac{1}{4}} = \mathrm{H}_0^1$ for example, we have $(\tilde{S}_\tau u)(0) = (\tilde{S}_\tau u)(\pi) = 0$, so that only the second set of boundary conditions needs to be checked. For this, writing S_τ^0 for the semigroup generated by Δ_0^2, note that we have the identity

$$\tilde{S}_\tau u = S_\tau^0 u + a\int_0^\tau S_{\tau-r}^0 A\tilde{S}_\tau u\, dr + a\int_0^\tau S_{\tau-r}^0 A\big(\tilde{S}_r u - \tilde{S}_\tau u\big)\, dr. \tag{12}$$

Therefore, the first term in this equation belongs to H_0^4. Furthermore, it follows from the definition of A that the H_0^4 norm of the third term is bounded by $C\int_0^\tau (\tau - r)^{-1-\frac{3}{8}-\varepsilon}\|\tilde{S}_r u - \tilde{S}_\tau u\|_{\mathrm{H}_0^2}\, dr$. Since we know already that $\mathrm{H}_0^2 = \tilde{\mathcal{H}}^{\frac{1}{2}}$, and since $\tilde{S}_\tau u \in \tilde{\mathcal{H}}^\alpha$ for every $\alpha > 0$, it follows from standard analytic semigroup theory that $\|\tilde{S}_r u - \tilde{S}_\tau u\|_{\mathrm{H}_0^2} \le C|r - \tau|$. So that the third term in (12) also belongs to H_0^4. The second term can be rewritten as

$$\int_0^\tau S_{\tau-r}^0 A\tilde{S}_\tau u\, dr = -\Delta_0^{-2} A\tilde{S}_\tau u - S_\tau^0 \int_0^\infty S_r^0 A\tilde{S}_\tau u\, dr.$$

Collecting all of this, we conclude that we can write

$$\tilde{S}_\tau u = -a\Delta_0^{-2} A\tilde{S}_\tau u + R_\tau u,$$

where $R_\tau u \in \mathrm{H}_0^4$. On the other hand, using an approximation argument, one can check that if $f \in \mathrm{C}^1$, then $g = \Delta_0^{-2} Af$ satisfies the boundary conditions $g''(0) = f'(0)$ and $g''(\pi) = -f'(\pi)$, from which the claim follows at once. ∎

Corollary 13. *For every $\alpha \in (-\frac{1}{8}, \frac{1}{8})$, we have the identity $\mathcal{H}_\alpha = \mathrm{H}^{4\alpha}\big([0, \pi], \mathbb{R}^d\big)$. For every $\alpha \in (\frac{1}{8}, \frac{1}{2}]$, we have the identity $\mathcal{H}_\alpha = \mathrm{H}^{4\alpha}\big([0, \pi], \mathbb{R}^d\big) \cap \mathrm{C}_0\big([0, \pi], \mathbb{R}^d\big)$, where $\mathrm{C}_0\big([0, \pi], \mathbb{R}^d\big)$ denotes the set of continuous functions vanishing at their endpoints. For every $\alpha \in [-\frac{1}{2}, -\frac{1}{8})$, we have $\mathcal{H}_\alpha = \mathrm{H}^{4\alpha}\big([0, \pi], \mathbb{R}^d\big)/\sim$, where the relation $\sim$ identifies distributions that differ only by a linear combination of δ_0 and δ_π.*

Proof. By Proposition 12, we already know that $\mathcal{H}_\alpha = \mathrm{H}_0^{4\alpha}$ for $\alpha \in [0, \frac{1}{2}]$. The claim for $\alpha < \frac{1}{8}$ then follows from Lemma 10 while the claim for $\alpha \in (\frac{1}{8}, \frac{1}{2})$ follows from Lemma 11. The remaining claims follow from duality. ∎

3.3 Well-Behaved Projection Operators

We will later identify the stationary distribution of the SPDE (9) by using a finite dimensional approximation argument. When projecting the equation to a finite dimensional subspace, the most natural choice of a projection would be to use the orthogonal projection Π_n onto the space

spanned by the first n eigenfunctions of $\bar{\mathcal{L}}$, but it transpires that these projections don't possess enough regularity. Instead, we will need to use, in some places, the operators $\hat{\Pi}_n$ given by

$$\hat{\Pi}_n x = \sum_{k=1}^{n} \frac{n-k}{n} \langle x, e_k \rangle e_k, \tag{13}$$

where the e_k are the eigenfunctions of $\bar{\mathcal{L}}$. The purpose of this section is to prove the required regularity properties for $\hat{\Pi}_n$.

We use the Hölder norms

$$\|x\|_{\mathrm{C}^{1+\alpha}} = \begin{cases} \|x\|_\infty + \|\dot{x}\|_\infty + \sup_{s \neq t} \dfrac{|\dot{x}(t) - \dot{x}(s)|}{|t-s|^\alpha} & \text{if } x \in \mathrm{C}^1 \text{ and} \\ +\infty & \text{else} \end{cases}$$

where $\alpha \in [0,1)$ and write $\mathrm{C}^{1+\alpha} = \bigl\{ x \in \mathrm{C}^1 \bigm| \|x\|_{\mathrm{C}^{1+\alpha}} < \infty \bigr\}$ and $\mathrm{C}_0^{1+\alpha} = \bigl\{ x \in \mathrm{C}^{1+\alpha} \bigm| x(0) = x(\pi) = 0 \bigr\}$.

Lemma 14. *Let $f_k \colon [0,\pi] \to \mathbb{R}$ be defined by $f_k(t) = \sin(kt)$. Define $\hat{\Pi}_n^0 x = \sum_{k=1}^n \frac{n-k}{n} \langle x, f_k \rangle f_k$.*

a) Let F_n be the Fejér kernel given by $F_n(t) = \frac{1}{n}\bigl(\sin(\frac{nt}{2})/\sin(\frac{t}{2})\bigr)^2$ for all $t \in [-\pi,\pi]$. Then $\hat{\Pi}_n^0 x = -\frac{1}{4} F_n * \tilde{x}$, where $\tilde{x}$ is the antisymmetric continuation of x.

b) $\|\hat{\Pi}_n^0 x\|_{\mathrm{C}^{1+\alpha}} \leq 2\pi \|x\|_{\mathrm{C}^{1+\alpha}}$ for all $x \in \mathrm{C}^{1+\alpha}$ and all $\alpha \in (0,1)$.

Proof. a) Since $\int_{-\pi}^{\pi} \tilde{x}(t) \sin(kt)\, dt = 2\langle x, f_k \rangle$ and $\int_{-\pi}^{\pi} \tilde{x}(t) \cos(kt)\, dt = 0$, it follows from trigonometric identities that

$$\langle x, f_k \rangle f_k(s) = -\frac{1}{2} \int_{-\pi}^{\pi} \tilde{x}(t) \cos\bigl(k(s-t)\bigr)\, dt.$$

The result then follows from the fact that $F_n(t) = 2\sum_{k=1}^n \frac{n-k}{n} \cos kt$.

b) This follows directly from part a, using the definition of the $\mathrm{C}^{1+\alpha}$-norm and properties of the convolution operator. ∎

Lemma 15. *Let $\alpha \in (0,1)$ and let $x \in \mathrm{C}^{1+\alpha}$ with $x(0) = x(\pi) = 0$. Then there exists a constant $c > 0$ such that the bounds*

1) $\|f_k\|_{\mathrm{C}^{1+\alpha}} \leq c\, k^{1+\alpha}$,
2) $\|e_k - f_k\|_{\mathrm{C}^{1+\alpha}} \leq c\, k^{\alpha}$,
3) $|\langle x, f_k \rangle| \leq c \|x\|_{\mathrm{C}^{1+\alpha}} k^{-1-\alpha}$, and
4) $|\langle x, e_k - f_k \rangle| \leq c \|x\|_{\mathrm{C}^{1+\alpha}} k^{-2-\alpha}$

hold for every $k \in \mathbb{N}$.

Proof. The first bound is standard. The second bound follows immediately from lemma 9, part b. For the third bound, we use partial integration to get

$$\langle x, f_k \rangle = \frac{1}{k} \int_0^\pi \dot{x}(t) \cos(kt)\, dt = \frac{1}{k} \sum_{j=0}^{k-1} (-1)^j \int_0^{\pi/k} \dot{x}\bigl(t + \frac{j}{k}\pi\bigr) \cos(ks)\, ds.$$

Writing $|\dot{x}|_\alpha = \sup_{s \neq t} \frac{|\dot{x}(t) - \dot{x}(s)|}{|t-s|^\alpha}$, it is easy to see that each term of the sum is of order $\mathcal{O}\bigl(|\dot{x}|_\alpha / k^{1+\alpha}\bigr)$ and the claim follows from this.

The bound on $\langle x, e_k - f_k \rangle$ follows similarly: if g is any $\mathrm{C}^{1+\alpha}$ function with $g(0) = g(\pi) = 0$, we can use integration by parts to get

$$\int_0^\pi g(t)\, \sin(kt)\, dt = \frac{1}{k} \int_0^\pi \dot{g}(t) \cos(kt)\, dt,$$
$$\int_0^\pi g(t)\, e^{-kt}\, dt = \frac{1}{k} \int_0^\pi \dot{g}(t) e^{-kt}\, dt,$$

and similar results for integrals against $\cos kt$ and $e^{-k(\pi - t)}$. As above, these expressions are bounded by $\mathcal{O}(k^{-1-\alpha})$. The claim now follows from lemma 9, part b, by absorbing the slowly varying terms $g_k^{(j)}$ into g. ∎

The following lemma collects all the properties we will require for the operators $\hat{\Pi}_n$. These will be used in the proof of propositions 26, below.

Lemma 16. *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of eigenfunctions of $\mathcal{L}$ and denote by Π_n the orthogonal projection of $\mathcal{H}$ onto $E_n = \operatorname{span}\{e_1, \dots, e_n\}$. Define $\hat{\Pi}_n \colon \mathcal{H} \to E_n$ as in (13). Then the following statements hold:*

a) $\hat{\Pi}_n \circ \Pi_n = \hat{\Pi}_n$.

b) $\hat{\Pi}_n x \to x$ in $\mathcal{H}_\alpha$ as $n \to \infty$ for all $\alpha \in \mathbb{R}$.

c) $\|\hat{\Pi}_n\|_{\mathcal{H}_\alpha} \le 1$ for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.

d) For every $0 < \alpha < \beta < 1$ we have $\|\hat{\Pi}_n\|_{\mathrm{C}_0^{1+\beta} \to \mathrm{C}_0^{1+\alpha}} < \infty$.

e) Let $0 < \alpha < \beta < 1/2$ and $x \in \mathrm{C}_0^{1+\beta}$. Then $\|\hat{\Pi}_n x - x\|_{\mathrm{C}^{1+\alpha}} \to 0$ as $n \to \infty$.

Proof. Statement a is clear from the definition of $\hat{\Pi}_n$. Let $x = \sum_{k=1}^{\infty} x_k e_k \in \mathcal{H}_\alpha$. Then

$$x - \hat{\Pi}_n x = \sum_{k=1}^{n} \frac{k}{n} x_k e_k + \sum_{k=n+1}^{\infty} x_k e_k$$

and thus, writing λ_k for the eigenvalues of $-\mathcal{L}$,

$$\big\|x - \hat{\Pi}_n x\big\|_{\mathcal{H}_\alpha}^2 = \big\|(-\mathcal{L})^\alpha (x - \hat{\Pi}_n x)\big\|_{\mathrm{L}^2}^2 = \sum_{k=1}^{n} \frac{k^2}{n^2} \lambda_k^{2\alpha} x_k^2 + \sum_{k=n+1}^{\infty} \lambda_k^{2\alpha} x_k^2 \longrightarrow 0$$

as $n \to \infty$. This proves statement b. Similarly, we have

$$\|\hat{\Pi}_n x\|_{\mathcal{H}_\alpha}^2 = \sum_{k=1}^{n} \frac{(n-k)^2}{n^2} \lambda_k^{2\alpha} x_k^2 \le \sum_{k=1}^{\infty} \lambda_k^{2\alpha} x_k^2 = \|x\|_{\mathcal{H}_\alpha}^2$$

which is statement c.

d) From lemma 15 we get $\|e_k\|_{\mathrm{C}^{1+\alpha}} \le \|f_k\|_{\mathrm{C}^{1+\alpha}} + \|e_k - f_k\|_{\mathrm{C}^{1+\alpha}} \le c\, k^{1+\alpha}$. Using this and the other bounds from lemma 15, we obtain

$$\begin{aligned}
\|\hat{\Pi}_n^0 x - \hat{\Pi} x\|_{\mathrm{C}^{1+\alpha}} &= \sum_{k=1}^{n} \frac{n-k}{n} \big\|\langle x, e_k - f_k\rangle e_k + \langle x, f_k\rangle (e_k - f_k)\big\|_{\mathrm{C}^{1+\alpha}} \\
&\le \sum_{k=1}^{\infty} \Big(\big|\langle x, e_k - f_k\rangle\big| \|e_k\|_{\mathrm{C}^{1+\alpha}} + \big|\langle x, f_k\rangle\big| \|e_k - f_k\|_{\mathrm{C}^{1+\alpha}} \Big) \\
&\le C \|x\|_{\mathrm{C}^{1+\beta}} \sum_{k=1}^{\infty} k^{-1-(\beta-\alpha)}.
\end{aligned}$$

Since we already know that $\hat{\Pi}_n^0$ satisfies the requested bound, the claim follows.

e) Let $\varepsilon > 0$. We can write $x \in \mathrm{C}_0^{1+\beta}$ as $x = y + z$ with $\|y\|_{\mathrm{C}^{1+\alpha}} \le \varepsilon$ and $z \in \mathrm{H}^2$ with $z(0) = z(T) = 0$. This gives

$$\big\|\hat{\Pi}_n x - x\big\|_{\mathrm{C}^{1+\alpha}} \le c\varepsilon + \big\|\hat{\Pi}_n z - z\big\|_{\mathrm{C}^{1+\alpha}}.$$

Because $1 + \alpha + 1/2 < 2$, we have $\|z\|_{\mathrm{C}^{1+\alpha}} \le c\|z\|_{\mathrm{H}^2}$ for all $z \in \mathrm{H}^2$. Corollary 13 gives $\mathrm{H}^2 \cap C_0 = \mathcal{H}_{1/2}$ and thus

$$\big\|\hat{\Pi}_n x - x\big\|_{\mathrm{C}^{1+\alpha}} \le c\varepsilon + \big\|\hat{\Pi}_n z - z\big\|_{\mathcal{H}_{1/2}} \to c\varepsilon$$

as $n \to \infty$ by part b. Since we can choose $\varepsilon > 0$ arbitrarily small, this completes the proof. ∎

4 The Linear Case

This section gives the proof of theorem 4 for the linear case $f \equiv 0$.

Lemma 17. *Let $\mathcal{L}$ and $\bar{x}$ be given by definition 3. Then $Q_0^{0,x-;T,x+} = \mathcal{N}\big(\bar{x}, -\mathcal{L}^{-1}\big)$.*

Proof. Since, for $f = 0$, the components of the solution of (6) are independent, it suffices to work in dimension $d = 1$. First consider the unconditioned process, described by (6). It is easy to check that p satisfies

$$p(t) = \mathrm{e}^{-t/2m} p(0) + \sqrt{\frac{1}{2}} \int_0^t \mathrm{e}^{-(t-r)/2m}\, dw(r)$$

and thus

$$q(t) = x_- + 2\big(1 - \mathrm{e}^{-t/2m}\big)p(0) + \sqrt{\frac{1}{2m^2}} \int_0^t \int_0^v \mathrm{e}^{-(v-r)/2m}\, dw(r)\, dv.$$

The mean of this process is

$$\bar{x}_0(t) = \mathbb{E}\big(q(t)\big) = x_-$$

and, since $p(0)$ is independent of w, the covariance function can be found as

$$\begin{aligned} C_0(s,t) &= \mathrm{Cov}\big(q(s), q(t)\big) \\ &= 4\big(1 - \mathrm{e}^{-s/2m}\big)\big(1 - \mathrm{e}^{-t/2m}\big)\frac{m}{2} + \frac{1}{2m^2} \int_0^s \int_0^t \int_0^{u\wedge v} \mathrm{e}^{-(u+v-2r)/2m}\, dr\, dv\, du \end{aligned}$$

for all $s, t \in [0, T]$. Evaluating the integrals and combining the resulting terms allows to simplify this to

$$C_0(s,t) = 2\big(s \wedge t\big) + 2m\big(\mathrm{e}^{-s/2m} + \mathrm{e}^{-t/2m} - \mathrm{e}^{-|s-t|/2m} - 1\big).$$

Denote the mean and covariance function of the process conditioned on $q(T) = x_+$ by $\bar{x}$ and C, respectively. From [HSVW05, equations (3.15) and (3.16)] we know that

$$\bar{x}(t) = \bar{x}_0(t) + C_0(t,T)C_0(T,T)^{-1}\big(x_+ - x_-\big)$$

and

$$C(s,t) = C_0(s,t) - C_0(s,T)C_0(T,T)^{-1}C_0(T,t)$$

for all $s, t \in [0, T]$. The covariance operator of $Q_0^{0,x_-;T,x_+}$ is then given by

$$\mathcal{C}f(s) = \int_0^T C(s,t)f(t)\, dt.$$

To complete the proof we have to verify that $\bar{x}$ and $\mathcal{C}$ have the required form. The following facts are easily checked:

(i) the first the derivatives $C_0(s,t)$, $\partial_t C_0(s,t)$ and $\partial_t^2 C_0(s,t)$ are continuous at $t = s$ and the third derivative at $t = s$ jumps according to

$$\partial_t^3 C_0(s, s+) - \partial_t^3 C_0(s, s-) = \frac{1}{2m^2};$$

(ii) the derivative boundary conditions

$$2m\, \partial_t^2 C_0(s,0) = \partial_t C_0(s,0), \quad 2m\, \partial_t^2 C_0(s,T) = -\partial_t C_0(s,T)$$

are satisfied;

(iii) the left boundary condition

$$C_0(s,0) = 0$$

holds; and

(iv) $LC_0(T,t) = 0$.

Clearly, by (ii) and (iii), the mean $\bar{x}$ satisfies the required boundary conditions (8) and by (iv) it also satisfies $L\bar{x} = 0$. From the definition of L and in particular from properties (i) and (iv) we can deduce

$$-LC(s,t) = \delta(t-s)$$

and using (ii) and (iii) we deduce that $C(t,s)$ satisfies the boundary conditions (7). Thus C is the Green's function of $-\mathcal{L}$ and we can deduce that $\mathcal{C} = -\mathcal{L}^{-1}$ as required. ∎

Proposition 18. *Consider the* $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$*-valued equation*

$$dy(\tau) = \mathcal{L}(y(\tau) - \bar{x})\,d\tau + \sqrt{2}\,dw(\tau), \qquad y(0) = y_0 \tag{14}$$

where $y_0 \in \mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$*. Then the following statements hold:*

a) Equation (14) *has a unique, global, continuous mild solution.*

b) For every $\alpha < 3/8$ *the solution* y *is a.s. continuous with values in* $\mathcal{H}_\alpha$*.*

c) The distribution $Q_0^{0,x_-;T,x_+}$ *is the unique stationary distribution for* (9)*.*

Proof. From lemma 17 we know $Q_0^{0,x_-;T,x_+} = \mathcal{N}\big(\bar{x}, (-\mathcal{L})^{-1}\big)$. Thus we can apply [HSVW05, lemma 2.2] to get that (14) has a continuous, $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$-valued mild solution and ν is its unique stationary distribution.

Let λ_k, $k \in \mathbb{N}$ be the eigenvalues of $-\mathcal{L}$. Then, using lemma 9, $\operatorname{tr}(-\mathcal{L})^{-2\beta} = \sum_{k\in\mathbb{N}} \lambda_k^{-2\beta} < \infty$ if and only if $\beta > 1/8$. Thus, for example by applying [Hai09, Theorem 5.13] to $y - \bar{x}$, the solution takes values in $\mathcal{H}_\alpha$ for every $\alpha < 1/2 - 1/8 = 3/8$ and is continuous by [Hai09, Theorem 5.17] (see also [DPZ92] for very similar results). This completes the proof. ∎

The regularity of the solution given in proposition 18 is consistent with the regularity of the target distribution $Q_0^{0,x_-;T,x_+}$: the process $\dot{x}$ in (6) is continuous and lives in $\mathrm{H}^{1/2-\varepsilon}$ and thus x is in $\mathrm{H}^{3/2-\varepsilon}$ for all $\varepsilon > 0$. On the other hand, corollary 13 shows that $\mathcal{H}_\alpha \subseteq \mathrm{H}^{4\alpha}$ and thus that y also takes values in $\mathrm{H}^{3/2-\varepsilon}$ for all $\varepsilon > 0$. The following lemma provides an additional regularity result for x in stationarity.

Lemma 19. *Let* $\alpha < 1/2$*. Then* $x \in C^{1+\alpha}$ *for* $Q_0^{0,x_-;T,x_+}$*-almost all* x*.*

Proof. The result is a direct consequence of [Hai09, Corollary 3.22]: Let e_k be the eigenfunctions of $-\mathcal{L}$ with corresponding eigenvalues λ_k. By lemma 17, part b, the random variable

$$X + \bar{x} = \sum_{k=1}^{\infty} \frac{\eta_k}{\sqrt{\lambda_k}} e_k,$$

where the η_k are i.i.d. standard Gaussian random variables, has distribution $Q_0^{0,x_-;T,x_+}$. We have to show that the derivative

$$X' + \bar{x}' = \sum_{k=1}^{\infty} \frac{\eta_k}{\sqrt{\lambda_k}} e_k'$$

is α-Hölder continuous.

Let $\delta \in (2\alpha, 1)$. By lemma 9 we have $\|e_k'\|_\infty = \mathcal{O}(k)$, $\|e_k''\|_\infty = \mathcal{O}(k^2)$ and $\lambda_k = ck^4 + \mathcal{O}(k^2)$ for some $c > 0$. This gives

$$S_1^2 = \sum_{k=1}^{\infty} \Big\| \frac{e_k'}{\sqrt{\lambda_k}} \Big\|_\infty^2 < \infty, \qquad S_2^2 = \sum_{k=1}^{\infty} \Big\| \frac{e_k'}{\sqrt{\lambda_k}} \Big\|_\infty^{2-\delta} \operatorname{Lip}\big(\frac{e_k'}{\sqrt{\lambda_k}}\big)^\delta \le \sum_{k=1}^{\infty} \frac{c}{k^{2-\delta}} < \infty.$$

Thus the conditions of [Hai09, Corollary 3.22] are satisfied, and we get the required Hölder continuity. ∎

5 The Nonlinear Case

In this section we complete the proof of theorem 4. The proof is split in a sequence of results which identify the target distribution $Q_f^{0,x_-;T,x_+}$, determine the regularity properties of the drift $\mathcal{N}$, give existence of global solutions to the SDE (9), and finally identify the stationary distribution of this equation.

Lemma 20. *Assume that* $f\colon \mathbb{R}^d \to \mathbb{R}^d$ *is such that the SDE* (6) *a.s. has a solution up to time* T*. Let* $\mu = Q_f^{0,x_-;T,x_+}$ *and* $\nu = Q_0^{0,x_-;T,x_+}$ *be the distributions on* $\mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$ *from definition 2. Then the density* $\varphi = \frac{d\mu}{d\nu}$ *is given by*

$$\varphi(x) = \frac{1}{Z} \exp\Big(m\langle f(x_+), \dot{x}(T)\rangle - m\langle f(x_-), \dot{x}(0)\rangle$$

$$- \int_0^T m\langle Df\big(x(t)\big)\dot{x}(t), \dot{x}(t)\rangle - \langle f\big(x(t)\big), \dot{x}(t)\rangle + \frac{1}{2}|f(x(t))|^2\, dt\Big)$$

where Df is the Jacobian of f and Z is the required normalisation constant.

Proof. Let $\tilde{\mu}_{\dot{x}} = P_f^{0,x_-}$ be the unconditioned distribution of $\dot{x}$ in (6) and let $\tilde{\nu}_{\dot{x}} = P_0^{0,x_-}$ the same distribution, but for $f = 0$. Then the Girsanov formula, *e.g.* in the form of [HSV07, lemma 9], gives the density of $\tilde{\mu}_{\dot{x}}$ w.r.t. $\tilde{\nu}_{\dot{x}}$:

$$\frac{d\tilde{\mu}_{\dot{x}}}{d\tilde{\nu}_{\dot{x}}}(\dot{x}) = \exp\Big(\int_0^T \langle f\big(x(t)\big), d\dot{x}(t)\rangle + \frac{1}{2}\int_0^T \langle f\big(x(t)\big), \frac{1}{m}\dot{x}(t) - f\big(x(t)\big)\rangle\, dt\Big)$$

where x is a deterministic function of $\dot{x}$ via the relation $x(t) = x_- + \int_0^t \dot{x}(s)\, ds$. Since $t \mapsto f\big(x(t)\big)$ has bounded variation, we can use partial integration to get

$$\int_0^T \langle f\big(x(t)\big), d\dot{x}(t)\rangle = \langle f\big(x(T)\big), \dot{x}(T)\rangle - \langle f\big(x(0)\big), \dot{x}(0)\rangle - \int_0^T \langle \dot{x}(t), Df\big(x(t)\big)\dot{x}(t)\rangle\, dt.$$

Substituting this expression into the formula for $d\tilde{\mu}_{\dot{x}}/d\tilde{\nu}_{\dot{x}}$ and using substitution to switch from $\dot{x}$ to x gives

$$\begin{aligned}\frac{d\tilde{\mu}_x}{d\tilde{\nu}_x}(x) = \exp\Big(& m\langle f\big(x(T)\big), \dot{x}(T)\rangle - m\langle f\big(x(0)\big), \dot{x}(0)\rangle \\ & - m\int_0^T \langle \dot{x}(t), Df\big(x(t)\big)\dot{x}(t)\rangle\, dt + \frac{1}{2}\int_0^T \langle f\big(x(t)\big), \dot{x}(t)\rangle - |f(x(t))|^2\, dt\Big)\end{aligned}$$

where $\tilde{\mu}_x = Q_f^{0,x_-}$ is the unconditioned distribution of x in (6) and $\tilde{\nu}_x = Q_0^{0,x_-}$ is the corresponding distribution for $f = 0$. Now we can condition on $x(T) = x_+$, *e.g.* using [HSV07, lemma 5.3] to get the result. ∎

Lemma 21. *Let $\alpha \in [0, 1)$. Then there is a $c > 0$ such that*

$$\|\dot{x}\|_\infty^{1+\alpha} \le c\|x\|_\infty^\alpha \|x\|_{\mathrm{C}^{1+\alpha}}$$

for all $x \in \mathrm{C}^1\big([0,T], \mathbb{R}^d\big)$.

Proof. The claim for $\alpha = 0$ is trivial, so we can assume $\alpha \neq 0$. Assume first the case $d = 1$. Write $|\dot{x}|_\alpha = \sup_{s\neq t} \frac{|\dot{x}(t)-\dot{x}(s)|}{|t-s|^\alpha}$ and let $t \in [0, T]$ such that $|\dot{x}(t)| = \|\dot{x}\|_\infty$. Then

$$|\dot{x}|_\alpha \ge \frac{|\dot{x}(t) - \dot{x}(s)|}{|t-s|^\alpha} \ge \frac{\|\dot{x}\|_\infty - |\dot{x}(s)|}{|t-s|^\alpha}$$

and thus $|\dot{x}(s)| \ge \|\dot{x}\|_\infty - |t-s|^\alpha |\dot{x}|_\alpha$ for all $s \in [0, T]$. This allows to conclude that $|\dot{x}(t)| \ge \frac{1}{2}\|\dot{x}\|_\infty$ on an interval of length at least $T \wedge \|\dot{x}\|_\infty^{1/\alpha}/(2|\dot{x}|_\alpha)^{1/\alpha}$. Since we assumed $d = 1$, this gives

$$\|x\|_\infty \ge \frac{1}{2}\cdot\frac{1}{2}\|\dot{x}\|_\infty \cdot \Big(T \wedge \frac{\|\dot{x}\|_\infty^{1/\alpha}}{2^{1/\alpha}|\dot{x}|_\alpha^{1/\alpha}}\Big) = \min\Big(\frac{T}{4}\|\dot{x}\|_\infty, \frac{\|\dot{x}\|_\infty^{1+1/\alpha}}{2^{2+1/\alpha}|\dot{x}|_\alpha^{1/\alpha}}\Big)$$

and by solving this inequality for $\|\dot{x}\|_\infty$ we find

$$\|\dot{x}\|_\infty^{1+\alpha} \le c\|x\|_\infty^\alpha \max\big(|\dot{x}|_\alpha, \|x\|_\infty\big) \le c\|x\|_\infty^\alpha \|x\|_{\mathrm{C}^{1+\alpha}}$$

for some constant c.

For $d > 1$ we apply the inequality componentwise: Since, for $z \in \mathbb{R}^d$, we have $\|z\|_2/\sqrt{d} \le \|z\|_\infty \le \|z\|_2$, we get

$$\begin{aligned}\|\dot{x}\|_\infty^{1+\alpha} &\le c \max_{j=1,\dots,d}\Big(\|x_j\|_\infty^\alpha\big(\|x_j\|_\infty + \|\dot{x}_j\|_\infty + |x_j|_\alpha\big)\Big) \\ &\le c\|x\|_\infty^\alpha\big(\|x\|_\infty + \|\dot{x}\|_\infty + |\dot{x}|_\alpha\big)\end{aligned}$$

where c is increased as needed. This completes the proof. ∎

The following bound for the density φ will be used to show that the stationary distributions of approximations for the sampling SPDE (9) are uniformly integrable.

Lemma 22. *Let φ be the density from lemma 20, $U = \log\varphi$ and $\nu = Q_0^{0,x_-;T,x_+}$ and $\alpha \in (0,1)$. Then for every $\varepsilon > 0$ there is an $M > 0$ such that for ν-almost all x we have*

$$U(x) \le \varepsilon\|x\|^2_{\mathrm{C}^{1+\alpha}} + M.$$

Proof. We bound the five terms in U one by one. For simplicity we denote all constants in the following estimates by the symbol c, the meaning of which changes from expression to expression.

1) Using the Cauchy-Schwarz inequality we get the bound

$$\langle f(x_+), \dot{x}(T)\rangle \le \big|f(x_+)\big|\|\dot{x}\|_\infty \le \big|f(x_+)\big|\|x\|_{\mathrm{C}^{1+\alpha}} \le \varepsilon\|x\|^2_{\mathrm{C}^{1+\alpha}} + c.$$

2) A very similar argument gives $-\langle f(x_-), \dot{x}(0)\rangle \le \varepsilon\|x\|^2_{\mathrm{C}^{1+\alpha}} + c$.
3) We can use Young's inequality together with lemma 21 to conclude that for every $\varepsilon > 0$ there is a $c > 0$ such that

$$\|\dot{x}\|_\infty \le \varepsilon\|x\|_{\mathrm{C}^{1+\alpha}} + c\|x\|_\infty$$

for all $x \in \mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$. Thus we have

$$\begin{aligned}
-\int_0^T \langle Df\big(x(t)\big)\dot{x}(t), \dot{x}(t)\rangle\, dt &\le \big\|\frac{d}{dt}f(x)\big\|_2 \,\big\|\dot{x}\big\|_2 \\
&\le T\big\|\frac{d}{dt}f(x)\big\|_\infty \,\big\|\dot{x}\big\|_\infty \\
&\le \Big(\varepsilon\|f(x)\|_{\mathrm{C}^{1+\alpha}} + c\|f(x)\|_\infty\Big)\Big(\varepsilon\|x\|_{\mathrm{C}^{1+\alpha}} + c\|x\|_\infty\Big).
\end{aligned}$$

Since f is differentiable with bounded derivatives we have $\|f(x)\|_{\mathrm{C}^{1+\alpha}} \le c\|x\|_{\mathrm{C}^{1+\alpha}} + c$ and by assumption there is a $\beta < 1$ such that $|f(x)| \le |x|^\beta + c$. Using these estimates we find

$$-\int_0^T \langle Df\big(x(t)\big)\dot{x}(t), \dot{x}(t)\rangle\, dt \le \Big(\varepsilon\|x\|_{\mathrm{C}^{1+\alpha}} + c\|x\|^\beta_\infty + c\Big)\Big(\varepsilon\|x\|_{\mathrm{C}^{1+\alpha}} + c\|x\|_\infty\Big)$$

and thus for every $\varepsilon > 0$ there is a $c > 0$ such that

$$-\int_0^T \langle Df\big(x(t)\big)\dot{x}(t), \dot{x}(t)\rangle\, dt \le \varepsilon\|x\|^2_{\mathrm{C}^{1+\alpha}} + \varepsilon\|x\|^2_\infty + c\|x\|^{1+\beta}_\infty + c.$$

Since $\beta < 1$, this gives the required bound.
4) Using the Cauchy-Schwarz inequality again, we get in a similar way

$$\int_0^T \langle f\big(x(t)\big), \dot{x}(t)\rangle\, dt \le \|f(x)\|_2\|\dot{x}\|_2 \le c\big(\|x\|^\beta_\infty + c\big)\|\dot{x}\|_\infty \le \varepsilon\|x\|^2_{\mathrm{C}^{1+\alpha}} + c.$$

5) Finally, we have $-\int_0^T \big|f(x(t))\big|^2\, dt < 0$.

Combining these bounds gives the required result. ∎

Lemma 23. *The drift $\mathcal{N}$ defined by (10) is locally Lipschitz from $\mathcal{H}_{1/4}$ to $\mathcal{H}_{-7/16}$. Furthermore, one can write $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3$ such that $\mathcal{N}_1$ does not depend on x and such that the bounds*

$$\|\mathcal{N}(x)\|_{\mathcal{H}_{-7/16}} \le c\big(1 + \|x\|^2_{\mathcal{H}_{1/4}}\big), \tag{15a}$$

$$\|\mathcal{N}_2(x) - \mathcal{N}_2(y)\|_{\mathcal{H}_{-5/16}} \le c\|x-y\|_{\mathcal{H}_{1/4}}\big(\|x\|_{\mathcal{H}_{1/4}} + \|y\|_{\mathcal{H}_{1/4}}\big), \tag{15b}$$

$$\|\mathcal{N}_3(x) - \mathcal{N}_3(y)\|_{\mathcal{H}_{-3/16}} \le c\|x-y\|_{\mathcal{H}_{1/4}}\big(\|x\|_{\mathcal{H}_{1/4}} + \|y\|_{\mathcal{H}_{1/4}}\big)^2 \tag{15c}$$

hold for all pairs $x, y \in \mathcal{H}_{1/4}$ and for some constant $c > 0$.

Proof. We use the characterisation of the spaces $\mathcal{H}_\alpha$ from corollary 13 and in particular the fact that if $x \in \mathcal{H}_{1/4}$, then x also belongs to H^1. Since we assumed that f_j and its derivatives up to the second order are globally Lipschitz, this implies that $f_j(x)$, $\partial_i f_j(x)$ and $\partial_{ij} f_k(x)$ all belong to H^1 and their norms are bounded by multiples of that of x.

Let now $x \in \mathrm{H}^1$. Then the following statements hold:

- $f_i(x)\partial_k f_i(x) \in \mathrm{H}^1$ since H^1 is stable under composition with smooth functions,
- $\partial_t x_i \big(\partial_i f_k(x) - \partial_k f_i(x)\big) \in \mathrm{L}^2$, for the same reason
- $\partial_t x_i \, \partial_t x_j \, \partial_{ij}^2 f_k(x) \in \mathrm{H}^t$ for all $t < -1/2$ since in this case $\mathrm{L}^1 \subseteq \mathrm{H}^t$ by Sobolev embedding,
- $\partial_t^2 x_j \big(\partial_j f_k(x) + \partial_k f_j(x)\big) \in \mathrm{H}^{-1}$ since H^{-1} is stable under multiplication by H^1-functions,
- $f_k(x_-)\partial_t \delta_0 \in \mathrm{H}^t$ and $f_k(x_+)\partial_t \delta_T \in \mathrm{H}^t$ for every $t < -3/2$.

It follows that $\mathcal{N}$ maps $\mathcal{H}_{1/4}$ into $\mathcal{H}_\alpha$ for every $\alpha < -\frac{3}{8}$. In particular, it maps $\mathcal{H}_{1/4}$ into $\mathcal{H}_{-\frac{7}{16}}$ as stated and the bound (15a) holds. We then define $\mathcal{N}_1$ as the term proportional to $f_k(x_-)\partial_t \delta_0 - f_k(x_+)\partial_t \delta_T$, $\mathcal{N}_3$ as the term proportional to $\partial_t x_i \, \partial_t x_j \, \partial_{ij}^2 f_k(x)$ and $\mathcal{N}_2$ as the sum of the remaining terms in the nonlinearity. With these definitions at hand, the bounds (15b) and (15c) follow easily. ∎

Proposition 24. *For every initial condition $x_0 \in \mathrm{L}^2\big([0,T], \mathbb{R}^d\big)$, the stochastic evolution equation (9) has a unique maximal local solution (x, τ^*). The solution satisfies $x(\tau) \in \mathcal{H}_{1/4}$ for every $\tau < \tau^*$ a.s. and $\sup_{\tau \uparrow \tau^*} \|x(\tau)\|_{\mathrm{L}^2} = \infty$ a.s. on the set $\{\tau^* < \infty\}$.*

Proof. Define

$$g(\tau) = S(\tau)x_0 + \sqrt{2} \int_0^\tau S(\tau - \sigma)\, dw(\sigma).$$

Let $R > U > 0$. For $x \colon (0, U] \to \mathcal{H}_{1/4}$ continuous, define

$$\|x\|_* = \sup_{\tau \in (0,U]} \tau^{1/4} \|x(\tau)\|_{\mathcal{H}_{1/4}},$$

and let $\mathcal{X}$ be the space of all such x with $x(0) = g(0)$ and $\|x\|_* < \infty$. Then $\big(\mathcal{X}, \|\cdot\|_*\big)$ is a Banach space. We find

$$\begin{aligned}
\|g\|_* &\le \sup_{\tau \in (0,U]} \tau^{1/4} \Big(\frac{1}{\tau^{1/4}} \|x_0\|_{\mathrm{L}^2} + \sqrt{2} \Big\| \int_0^\tau S(\tau - \sigma)\, dw(\sigma) \Big\|_{\mathcal{H}_{1/4}} \Big) \\
&\le \|x_0\|_{\mathrm{L}^2} + \sqrt{2} R \sup_{\tau \in [0,R]} \Big\| \int_0^\tau S(\tau - \sigma)\, dw(\sigma) \Big\|_{\mathcal{H}_{1/4}} \\
&=: \|x_0\|_{\mathrm{L}^2} + C_R
\end{aligned}$$

and thus $g \in \mathcal{X}$ for every $U < R$. Define a map $M_g \colon \mathcal{X} \to \mathcal{X}$ by

$$M_g x(\tau) = \int_0^\tau S(\tau - \sigma) \mathcal{N}\big(x_\sigma\big)\, d\sigma + g(\tau) \quad \forall \tau \in [0, U].$$

By the definition of a mild solution, local solutions up to time U coincide with the fixed points of this map.

Let $B(g, 1) \subseteq \mathcal{X}$ denote the closed ball around g with radius 1. By lemma 23, the nonlinearity $\mathcal{N} \colon \mathcal{H}_{1/4} \to \mathcal{H}_{-7/16}$ is locally Lipschitz and thus for all $x, y \in B(g, 1)$ we have

$$\begin{aligned}
\big\| M_g x - M_g y \big\|_* &\le \sup_{\tau \in (0,U]} \tau^{1/4} \int_0^\tau \big\| S(\tau - \sigma) \big(\mathcal{N}(x_\sigma) - \mathcal{N}(y_\sigma) \big) \big\|_{\mathcal{H}_{1/4}}\, d\sigma \\
&\le \sup_{\tau \in (0,U]} c\tau^{1/4} \int_0^\tau \Big(\frac{\big\| \mathcal{N}_2(x_\sigma) - \mathcal{N}_2(y_\sigma) \big\|_{\mathcal{H}_{-5/16}}}{(\tau - \sigma)^{9/16}} + \frac{\big\| \mathcal{N}_3(x_\sigma) - \mathcal{N}_3(y_\sigma) \big\|_{\mathcal{H}_{-3/16}}}{(\tau - \sigma)^{7/16}} \Big)\, d\sigma \\
&\le \sup_{\tau \in (0,U]} c\tau^{1/4} \int_0^\tau \big\| x_\sigma - y_\sigma \big\|_{\mathcal{H}_{1/4}} \Big(\frac{\|x_\sigma\|_{\mathcal{H}_{1/4}} + \|y_\sigma\|_{\mathcal{H}_{1/4}}}{(\tau - \sigma)^{9/16}} + \frac{\big(\|x_\sigma\|_{\mathcal{H}_{1/4}} + \|y_\sigma\|_{\mathcal{H}_{1/4}}\big)^2}{(\tau - \sigma)^{7/16}} \Big)\, d\sigma
\end{aligned}$$

$$\leq c\,U^{1/16}\,\|x-y\|_*\big(1+\|x\|_*+\|y\|_*\big)^2,$$

where c changes from line to line. Similarly, we have

$$\|M_g x - g\|_* \leq c\,U^{1/16}\,\|x\|_*^2 \leq c\,U^{1/16}\,\big(\|x-g\|_*+\|g\|_*\big)^2.$$

By choosing the final time U sufficiently small, we can then make sure that M_g is a contraction on the ball $B(g,1)$ and, by the Banach fixed point theorem, M_g has a unique fixed point. This gives a unique local solution of (9) up to time U.

By iterating this procedure, every time starting with the final point of the previously constructed segment, we obtain a solution up to a maximal time $\tau^* \leq R$. Since the length of each segment of this solution only depends on the L^2-norm of its starting point, we see that $\tau^* < R$ implies $\sup_{\tau<\tau^*}\|x(\tau)\|_{\mathrm{L}^2} = \infty$. Taking $R\to\infty$ completes the proof. ∎

Even if f is globally Lipschitz, the $\partial_t x_i\,\partial_t x_j\,\partial_{ij}^2 f_k$-term causes the nonlinearity $\mathcal{N}$ to be only locally Lipschitz. Thus, showing the existence of global solutions to the SDE (9) will need some care.

Proposition 25. *For every initial condition $x_0 \in \mathrm{L}^2\big([0,T],\mathbb{R}^d\big)$ the SPDE (9) has a unique global solution. For every $\tau>0$ the solution satisfies $\mathbb{E}\big(\|x(\tau)\|_{\mathrm{L}^2}^2\big)<\infty$.*

Proof. From proposition 24 we know that (9) has a local solution $(x,\tau_{\max})$. Let y be the solution of the linear SPDE from proposition 18, *i.e.*

$$y(\tau) = S(\tau)\big(x_0-\bar{x}\big) + \sqrt{2}\int_0^\tau S(\tau-\sigma)\,dw(\sigma) + \bar{x},$$

and define $z(\tau)=x(\tau)-y(\tau)$ for every $\tau\in[0,\tau_{\max})$. Then z satisfies the stochastic evolution equation

$$dz(\tau) = \mathcal{L}z(\tau)\,d\tau + \mathcal{N}\big(z(\tau)+y(\tau)\big)\,d\tau, \qquad z(0)=0.$$

Thus $\|z(\tau)\|_{\mathrm{L}^2}^2$ satisfies

$$\begin{aligned}\frac{d\|z(\tau)\|_{\mathrm{L}^2}^2}{d\tau} &= 2\big\langle z(\tau), \mathcal{L}z(\tau)+\mathcal{N}\big(z(\tau)+y(\tau)\big)\big\rangle \\ &= -4m^2\langle\partial_t^2 z(\tau),\partial_t^2 z(\tau)\rangle - 4m\big|\partial_t z(0)\big|^2 - 4m\big|\partial_t z(1)\big|^2 \\ &\qquad\qquad - \langle\partial_t z(\tau),\partial_t z(\tau)\rangle + 2\big\langle z(\tau),\mathcal{N}\big(z(\tau)+y(\tau)\big)\big\rangle \\ &\leq -c\|z(\tau)\|_{\mathrm{H}^2}^2 + 2\big\langle z(\tau),\mathcal{N}\big(z(\tau)+y(\tau)\big)\big\rangle\end{aligned} \tag{16}$$

for some $c>0$. This formal calculation can be made rigorous by a standard approximation argument, using for example Galerkin approximations.

We require *a priori* bounds of the form $\big\langle z,\mathcal{N}(z+y)\big\rangle \leq c\|z\|_{\mathrm{L}^2}^2 + \varepsilon\|z\|_{\mathrm{H}^2}^2 + c$ where $\varepsilon>0$ is small enough to be compensated by the negative $\|z\|_{\mathrm{H}^2}^2$-term in (16).

In order to obtain the required bounds we consider the five terms from the definition of $\mathcal{N}$ individually. For the purpose of these estimates we denote all numerical constants by $c>0$ and only track the y-dependency of the bounds explicitly. For the first term we get

$$\big\langle z_k, -f_i(z+y)\,\partial_k f_i(z+y)\big\rangle \leq c\|z\|_{\mathrm{L}^2}\|f(z+y)\|_{\mathrm{L}^2} \leq c\|z\|_{\mathrm{L}^2}^2 + c\|y\|_{\mathrm{L}^2}^2 + c.$$

For the second term we find

$$\begin{aligned}&\big\langle z_k, \partial_t(z_i+y_i)\,\partial_t(z_j+y_j)\,\partial_{ij}^2 f_k(z+y)\big\rangle \\ &\quad= \int_0^1 z_k\,\partial_t z_i\,\partial_t\big(\partial_i f_k(z+y)\big)\,dt + \int_0^1 z_k\,\partial_t y_i\,\partial_t(z_j+y_j)\,\partial_{ij}^2 f_k(z+y)\,dt \\ &\quad= -\int_0^1 \partial_t z_k\,\partial_t z_i\,\partial_i f_k(z+y)\,dt - \int_0^1 z_k\,\partial_t^2 z_i\,\partial_i f_k(z+y)\,dt \\ &\qquad\qquad + \int_0^1 z_k\,\partial_t y_i\,\partial_t(z_j+y_j)\,\partial_{ij}^2 f_k(z+y)\,dt\end{aligned}$$

$$\leq c\|z\|_{\mathrm{H}^1}^2 + c\|z\|_{\mathrm{L}^2}\|z\|_{\mathrm{H}^2} + c\|z\|_{\mathrm{L}^\infty}\|y\|_{\mathrm{H}^1}\big(\|z\|_{\mathrm{H}^1} + \|y\|_{\mathrm{H}^1}\big).$$

For the third term we have

$$\big\langle z_k, -\partial_t(z_i + y_i)\,(\partial_i f_k - \partial_k f_i)\big\rangle \leq c\|z\|_{\mathrm{L}^2}\big(\|z\|_{\mathrm{H}^1} + \|y\|_{\mathrm{H}^1}\big).$$

The fourth term can be bounded as

$$\begin{aligned}
&\big\langle z_k, \partial_t^2(z_i + y_i)\,(\partial_i f_k + \partial_k f_i)\big\rangle \\
&\quad= \big\langle z_k, \partial_t^2 z_i\,(\partial_i f_k + \partial_k f_i)\big\rangle \\
&\qquad\quad - \int_0^1 \partial_t z_k \partial_t y_i\,(\partial_i f_k + \partial_k f_i)\,dt - \int_0^1 z_k \partial_t y_i\,\partial_t(z_j + y_j)\,(\partial_{ij}^2 f_k + \partial_{jk}^2 f_i)\,dt \\
&\quad\leq c\|z\|_{\mathrm{L}^2}\|z\|_{\mathrm{H}^2} + c\|z\|_{\mathrm{H}^1}\|y\|_{\mathrm{H}^1} + c\|z\|_{\mathrm{L}^\infty}\|y\|_{\mathrm{H}^1}\big(\|z\|_{\mathrm{H}^1} + \|y\|_{\mathrm{H}^1}\big).
\end{aligned}$$

Finally, for the fifth term, involving the derivatives of Dirac distributions, we get

$$\big\langle z, \partial_t\delta_0\big\rangle = -z'(0) \leq c\|z\|_{\mathrm{H}^\alpha}, \qquad \big\langle z, -\partial_t\delta_1\big\rangle = z'(1) \leq c\|z\|_{\mathrm{H}^\alpha}$$

for every $\alpha > 3/2$.

To convert the bounds into the required form first note that for every $s \in (0,2)$ the interpolation inequality (see *e.g.* [Hai09, corollary 6.11]) gives $\|z\|_{\mathrm{H}^s}^2 \leq \|z\|_{\mathrm{L}^2}^{2-s}\|z\|_{\mathrm{H}^2}^s$ and, using Young's inequality, we can for every $\varepsilon > 0$ find a $c > 0$ such that

$$\|z\|_{\mathrm{L}^2}^{2-s}\|z\|_{\mathrm{H}^2}^s \leq c\|z\|_{\mathrm{L}^2}^2 + \varepsilon\|z\|_{\mathrm{H}^2}^2.$$

Using this relation we find a $c > 0$ such that

$$\|z\|_{\mathrm{L}^\infty}\|y\|_{\mathrm{H}^1}^2 \leq \frac{1}{2}\|z\|_{\mathrm{L}^\infty}^2 + \frac{1}{2}\|y\|_{\mathrm{H}^1}^4 \leq c\|z\|_{\mathrm{H}^1}^2 + c\|y\|_{\mathrm{H}^1}^4 \leq c\|z\|_{\mathrm{L}^2}^2 + \varepsilon\|z\|_{\mathrm{H}^2}^2 + c\|y\|_{\mathrm{H}^1}^4.$$

The terms of the form $\|z\|_{\mathrm{L}^\infty}\|z\|_{\mathrm{H}^1}\|y\|_{\mathrm{H}^1}$ can be bounded using the relation

$$\|z\|_{\mathrm{L}^\infty}\|z\|_{\mathrm{H}^1} \leq c\|z\|_{\mathrm{H}^{3/4}}\|z\|_{\mathrm{H}^1} \leq c\,\|z\|_{\mathrm{L}^2}^{5/8}\|z\|_{\mathrm{H}^2}^{3/8}\,\|z\|_{\mathrm{L}^2}^{1/2}\|z\|_{\mathrm{H}^2}^{1/2} = c\|z\|_{\mathrm{L}^2}^{9/8}\|z\|_{\mathrm{H}^2}^{7/8}.$$

Applying Young's inequality with $p = 16/7$ and $q = 16/9$ we find a $c > 0$ such that

$$\|z\|_{\mathrm{L}^\infty}\|z\|_{\mathrm{H}^1}\|y\|_{\mathrm{H}^1} \leq c\,\|z\|_{\mathrm{H}^2}^{7/8}\,\|z\|_{\mathrm{L}^2}^{9/8}\|y\|_{\mathrm{H}^1} \leq \varepsilon\|z\|_{\mathrm{H}^2}^2 + c\|z\|_{\mathrm{L}^2}^2\|y\|_{\mathrm{H}^1}^{16/9}.$$

Combining all these estimates, we find that for every $\varepsilon > 0$ there is a $c > 0$ such that

$$\big\langle z(\tau), \mathcal{N}\big(z(\tau) + y(\tau)\big)\big\rangle \leq c\big(1 + \|y\|_{\mathrm{H}^1}^{16/9}\big)\|z\|_{\mathrm{L}^2}^2 + \varepsilon\|z\|_{\mathrm{H}^2}^2 + c\big(1 + \|y\|_{\mathrm{L}^2}^2 + \|y\|_{\mathrm{H}^1}^4\big)$$

and substituting this bound into (16) for small enough $\varepsilon > 0$ we get

$$\frac{d\|z(\tau)\|_{\mathrm{L}^2}^2}{d\tau} \leq c\big(1 + \|y\|_{\mathrm{H}^1}^{16/9}\big)\big\|z(\tau)\big\|_{\mathrm{L}^2}^2 + c\big(1 + \|y\|_{\mathrm{L}^2}^2 + \|y\|_{\mathrm{H}^1}^4\big).$$

Gronwall's inequality gives

$$\begin{aligned}
\|z(\tau)\|_{\mathrm{L}^2}^2 \leq c\int_0^\tau \big(1 + \|y(\sigma)\|_{\mathrm{H}^1}^{16/9}\big) \int_0^\sigma \big(1 + \|y(r)\|_{\mathrm{L}^2}^2 + \|y(r)\|_{\mathrm{H}^1}^4\big)\,dr \\
\cdot \exp\Big(\int_\sigma^\tau \big(1 + \|y(r)\|_{\mathrm{H}^1}^{16/9}\big)\,dr\Big)\,d\sigma \qquad (17) \\
+ c\int_0^\tau \big(1 + \|y(\sigma)\|_{\mathrm{L}^2}^2 + \|y(\sigma)\|_{\mathrm{H}^1}^4\big)\,d\sigma.
\end{aligned}$$

Thus $\|z\|_{\mathrm{L}^2}$ cannot explode in finite time and from proposition 24 we get $\tau_{\max} = \infty$.

By proposition 18 we have $y \in \mathrm{L}^2\big([0,\tau], \mathrm{H}^1\big)$. Hence, by Fernique's theorem (see *e.g.* [Hai09, theorem 3.11]),

$$\mathbb{E}\Big(\exp\big(\varepsilon\int_0^\tau \|y(r)\|_{\mathrm{H}^1}^2\,dr\big)\Big) < \infty$$

for sufficiently small $\varepsilon > 0$. Thus, using the fact that $16/9 < 2$, we see that the right hand side of (17) has finite expectation for all $\tau > 0$. ∎

Now the only part of theorem 4 which we still need to prove is the statement about the stationary distribution of (9). This can be done using a finite dimensional approximation argument, similar to the proofs in [Zab88] and [HSV07, section 3]. Since these articles assumed that U was bounded from above and also assumed different regularity properties for the drift, the proof needs to be adapted for the situation here; to allow for easier reading, we include the full argument instead of just enumerating the required changes.

Proposition 26. *The distribution $Q_f^{0,x_-;T,x_+}$ is invariant for (9).*

Proof. Let φ be the density of $\mu = Q_f^{0,x_-;T,x_+}$ w.r.t. $\nu = Q_0^{0,x_-;T,x_+}$ as given by lemma 20 and let $U = \log\varphi$. Then we can compute the derivative of U at $x \in \mathcal{H}_{1/4}$ in direction $h \in \mathcal{H}_{7/16}$ as

$$\begin{aligned}\langle DU(x), h\rangle \;&=\; m\, f_k(x_+)\dot h_k(T) - m\, f_k(x_-)\dot h_k(0)\\ &\quad + 2\int_0^T \Big(-f_i\partial_k f_i + m\,\dot x_i\dot x_j\partial_{ij}^2 f_k - \frac{1}{2}\dot x_i(\partial_i f_k - \partial_k f_i) + m\ddot x_i(\partial_i f_k + \partial_k f_i)\Big)h_k(t)\,dt\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \;\langle \mathcal{N}(x), h\rangle.\end{aligned}$$

Here we used the fact that, by corollary 13, $h \in \mathcal{H}_{7/16}$ implies $h(0) = h(T) = 0$. This shows that the function U is Fréchet-differentiable with derivative $\mathcal{N}$. Let Π_n and $\hat\Pi_n$ be as in lemma 16 and define the approximations

$$\mathcal{N}_n = \big(U\circ\hat\Pi_n\big)' = \hat\Pi_n\mathcal{N}(\hat\Pi_n\,\cdot\,)$$

for $n \in \mathbb{N}$.

Consider the n-dimensional SDEs

$$dy_n(\tau) = \mathcal{L}y_n(\tau)\,d\tau \qquad\qquad + \sqrt{2}\Pi_n\,dw(\tau), \qquad y_n(0) = \Pi_n x_0$$

and

$$dx_n(\tau) = \mathcal{L}x_n(\tau)\,d\tau + \mathcal{N}_n\big(x_n(\tau)\big)\,d\tau + \sqrt{2}\Pi_n\,dw(\tau), \qquad x_n(0) = \Pi_n x_0.$$

Then, by finite dimensional results, the stationary distributions ν_n and μ_n of y_n and x_n respectively are given by

$$\nu_n = \nu\circ\Pi_n^{-1} \qquad\text{and}\qquad \frac{d\mu_n}{d\nu_n} = \exp(U\circ\hat\Pi_n).$$

Define the semigroup $(\mathcal{P}_\tau^n)_{\tau\ge0}$ on $C_{\mathrm{b}}(\mathcal{H},\mathbb{R})$ by $\mathcal{P}_\tau^n\varphi(x) = \mathbb{E}_x\big(\varphi(x_n(\tau))\big)$ for all $x \in E_n$ and $\varphi \in C_{\mathrm{b}}(\mathcal{H},\mathbb{R})$. Since the process x_n is μ_n-reversible, we have

$$\int_{\mathcal{H}} \varphi(x)\,\mathcal{P}_\tau^n\psi(x)\,d\mu_n(x) = \int_{\mathcal{H}} \psi(x)\,\mathcal{P}_\tau^n\varphi(x)\,d\mu_n(x). \tag{18}$$

for every $\varphi,\psi \in C_{\mathrm{b}}(\mathcal{H},\mathbb{R})$.

We need to find the limit of (18) as $n\to\infty$. For this, we first show that $x_n \to x$ in $\mathcal{H}_{1/4}$, uniformly on bounded time intervals: Let $U > 0$. Then we have

$$\begin{aligned}\big\|x_n(\tau) - x(\tau)\big\|_{\mathcal{H}_{1/4}} &\le \Big\|\big(\Pi_n - I\big)\Big(S(\tau)x_0 + \sqrt{2}\int_0^\tau S(\tau-\sigma)\,dW(\sigma)\Big)\Big\|_{\mathcal{H}_{1/4}}\\ &\quad + \Big\|\int_0^\tau S(\tau-\sigma)\big(\mathcal{N}_n(x(\sigma)) - \mathcal{N}(x(\sigma))\big)\,d\sigma\Big\|_{\mathcal{H}_{1/4}}\\ &\quad + \Big\|\int_0^\tau S(\tau-\sigma)\big(\mathcal{N}_n(x_n(\sigma)) - \mathcal{N}_n(x(\sigma))\big)\,d\sigma\Big\|_{\mathcal{H}_{1/4}}\\ &=: I_1(\tau) + I_2(\tau) + I_3(\tau)\end{aligned}$$

for all $\tau \in [0,U]$.

From the definition of $\|\cdot\|_{\mathcal{H}_\alpha}$ and the asymptotics of the eigenvalues of $\mathcal{L}$ in lemma 9 we get, for any $\beta > \alpha$, that there is a $c > 0$ such that the bound

$$\big\|\Pi_n x - x\big\|_{\mathcal{H}_\alpha} \le \frac{c}{n^{8(\beta-\alpha)}}\|x\|_{\mathcal{H}_\beta}$$

holds for all $x \in \mathcal{H}_\beta$ and all $n \in \mathbb{N}$. Let $\beta \in (1/4, 3/8)$. Then we know from proposition 18 that $\tau \mapsto S(\tau)x_0 + \sqrt{2}\int_0^\tau S(\tau-\sigma)\,dW(\sigma)$ is a continuous map from $[0,U]$ into $\mathcal{H}_\beta$. Combining these two statements, we find $\sup_{0\le\tau\le U} I_1(\tau) \to 0$ as $n \to \infty$.

From lemma 23 we know that $\mathcal{N}$ is locally Lipschitz from $\mathcal{H}_{1/4}$ to $\mathcal{H}_{-7/16}$. By lemma 16, part c, there is then a constant $K_r > 0$ such that

$$\big\|\mathcal{N}_n(x) - \mathcal{N}_n(y)\big\|_{\mathcal{H}_{-7/16}} \le K_r \|x-y\|_{\mathcal{H}_{1/4}}$$

for all $n \in \mathbb{N}$ and all x and y with $\|x\|_{\mathcal{H}_{1/4}}, \|y\|_{\mathcal{H}_{1/4}} \le r$. Thus, the $\mathcal{N}_n$ are also locally Lipschitz.

We can find $p, q > 1$ such that $p \cdot \frac{11}{16} < 1$ and $1/p + 1/q = 1$. For I_2 we get then

$$\begin{aligned} I_2(\tau) &\le \int_0^\tau \big\|S(\tau-\sigma)\big(\mathcal{N}_n(x(\sigma)) - \mathcal{N}(x(\sigma))\big)\big\|_{\mathcal{H}_{1/4}}\,d\sigma \\ &\le \int_0^\tau \big\|S(\tau-\sigma)\big\|_{\mathcal{H}_{-7/16}\to\mathcal{H}_{1/4}} \big\|\mathcal{N}_n(x(\sigma)) - \mathcal{N}(x(\sigma))\big\|_{\mathcal{H}_{-7/16}}\,d\sigma \\ &\le c\Big(\int_0^U \frac{1}{\sigma^{p\,11/16}}\,d\sigma\Big)^{1/p}\Big(\int_0^U \big\|\mathcal{N}_n(x(\sigma)) - \mathcal{N}(x(\sigma))\big\|^q_{\mathcal{H}_{-7/16}}\,d\sigma\Big)^{1/q}. \end{aligned}$$

The right hand side is independent of τ and converges to 0 as $n \to \infty$ by dominated convergence, using lemma 16, part b.

For $n \in \mathbb{N}$ define

$$T_{n,r} = \inf\Big\{\tau \in [0,U] \;\Big|\; \|x(\tau)\| > r \text{ or } \|x_n(\tau)\| > r\Big\},$$

with the convention $\inf\emptyset = U$. For $\tau \le T_{n,r}$ we have

$$I_3(\tau) \le K_r \int_0^\tau \big\|S(\tau-\sigma)\big\|_{\mathcal{H}_{-7/16}\to\mathcal{H}_{1/4}} \big\|x_n(\sigma) - x(\sigma)\big\|_{\mathcal{H}_{1/4}}\,d\sigma$$

and consequently

$$\begin{aligned} \big\|x_n(\tau) - x(\tau)\big\|_{\mathcal{H}_{1/4}} \le{}& \sup_{0\le\sigma\le U}\big(I_1(\sigma) + I_2(\sigma)\big) \\ &+ cK_r \int_0^\tau \frac{1}{(\tau-\sigma)^{11/16}} \big\|x_n(\sigma) - x(\sigma)\big\|\,d\sigma. \end{aligned}$$

Using Gronwall's lemma we can conclude

$$\big\|x_n(\tau) - x(\tau)\big\|_{\mathcal{H}_{1/4}} \le \sup_{0\le\sigma\le U}\big(I_1(\sigma) + I_2(\sigma)\big) \cdot \exp\Big(cK_r \int_0^U \frac{1}{\sigma^{11/16}}\,d\sigma\Big)$$

for all $\tau \le T_{n,r}$. As we have already seen, the right-hand side converges to 0 as $n \to \infty$.

Now choose $r > 0$ big enough such that $\sup_{0\le\tau\le U}\|x(\tau)\| \le r/4$. Then for sufficiently large n and all $\tau \le T_{n,r}$ we have $\|x_n(\tau) - x(\tau)\| \le r/4$ and thus $\sup_{0\le\tau\le T_{n,r}}\|x_n(\tau)\| \le r/2$. This implies $T_{n,r} = U$ for sufficiently large n. Thus we have $x_n \to x$ in $\mathrm{C}\big([0,U], \mathcal{H}_{1/4}\big)$ a.s.

Let $0 < \alpha < \beta < 1/2$. Define the semigroup $(\mathcal{P}_\tau)_{\tau\ge0}$ on $C_\mathrm{b}(\mathcal{H}, \mathbb{R})$ by $\mathcal{P}_\tau\varphi(x) = \mathbb{E}_x\big(\varphi(x(\tau))\big)$ for all $x \in \mathcal{H}$ and $\varphi \in C_\mathrm{b}(\mathcal{H}, \mathbb{R})$. Then, by dominated convergence, we have $\mathcal{P}_\tau^n\varphi(\Pi_n x) \to \mathcal{P}_\tau\varphi(x)$ as $n \to \infty$. By lemma 19, $x \in \mathrm{C}^{1+\beta}$ for ν-almost all x. Furthermore, $U\colon \mathrm{C}^{1+\alpha} \to \mathbb{R}$ is continuous and thus $U(\hat\Pi_n x) \to U(x)$ as $n \to \infty$ for ν-almost all x by lemma 16, part e.

Finally let $c = \|\hat\Pi_n\|_{\mathrm{C}_0^{1+\beta}\to\mathrm{C}_0^{1+\alpha}}$. Using Fernique's theorem we can choose $\varepsilon > 0$ such that the function $\exp\big(\varepsilon\, c\|x\|^2_{\mathrm{C}^{1+\beta}}\big)$ is ν-integrable. By lemma 22 we can find an $M > 0$ such that $U(\hat\Pi_n x) \le \varepsilon\|\hat\Pi_n x\|^2_{\mathrm{C}^{1+\alpha}} + M \le \varepsilon c\|x\|^2_{\mathrm{C}^{1+\beta}} + M$ for all $n \in \mathbb{N}$ and ν-almost all x. Then dominated convergence gives

$$\begin{aligned} \lim_{n\to\infty}\int_\mathcal{H} \varphi(x)\,\mathcal{P}_\tau^n\psi(x)\,d\mu_n(x) &= \lim_{n\to\infty}\int_\mathcal{H} \varphi(\Pi_n x)\,\mathcal{P}_\tau^n\psi(\Pi_n x)\,\mathrm{e}^{U(\hat\Pi_n x)}d\nu(x) \\ &= \int_\mathcal{H} \varphi(x)\,\mathcal{P}_\tau\psi(x)\,\mathrm{e}^{U(x)}d\nu(x) = \int_\mathcal{H} \varphi(x)\,\mathcal{P}_\tau\psi(x)\,d\mu(x) \end{aligned}$$

and using (18) we get

$$\int_{\mathcal{H}} \varphi(x)\, \mathcal{P}_\tau \psi(x)\, d\mu(x) = \int_{\mathcal{H}} \psi(x)\, \mathcal{P}_\tau \varphi(x)\, d\mu(x).$$

Thus the process x is μ-reversible which is the required result. ∎

Propositions 24, 25 and 26 together imply all claims of theorem 4 and so the proof of the result is complete.

References

[AHSV07] A. Apte, M. Hairer, A. M. Stuart and J. Voss. *Sampling The Posterior: An Approach to Non-Gaussian Data Assimilation.* Physica D: Nonlinear Phenomena, vol. 230, no. 1–2, pp. 50–64, 2007.

[BMPW01] D. Blömker, S. Maier-Paape and T. Wanner. *Spinodal decomposition for the Cahn-Hilliard-Cook equation.* Communications in Mathematical Physics, vol. 223, no. 3, pp. 553–582, 2001.

[DPD96] G. Da Prato and A. Debussche. *Stochastic Cahn-Hilliard equation.* Nonlinear Analysis A, vol. 26, no. 2, pp. 241–263, 1996.

[DPZ92] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 1992.

[Hai09] M. Hairer. *An Introduction to Stochastic PDEs.* Lecture notes, 2009. `http://arxiv.org/abs/0907.4178`

[Hai10] M. Hairer. *Singular Perturbations to Semilinear Stochastic Heat Equations*, 2010. Preprint available on arXiv. `http://arxiv.org/abs/1002.3722`

[HSV07] M. Hairer, A. M. Stuart and J. Voss. *Analysis of SPDEs Arising in Path Sampling, Part II: The Nonlinear Case.* Annals of Applied Probability, vol. 17, no. 5/6, pp. 1657–1706, 2007.

[HSV09] M. Hairer, A. M. Stuart and J. Voss. *Sampling Conditioned Diffusions.* In *Trends in Stochastic Analysis*, vol. 353 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 2009.

[HSVW05] M. Hairer, A. M. Stuart, J. Voss and P. Wiberg. *Analysis of SPDEs arising in Path Sampling, Part I: The Gaussian Case.* Communications in Mathematical Sciences, vol. 3, no. 4, pp. 587–603, 2005.

[Kat66] T. Kato. *Perturbation Theory for Linear Operators.* Springer, 1966.

[LCM96] K. B. Lauritsen, R. Cuerno and H. A. Makse. *Noisy Kuramoto-Sivashinsky equation for an erosion model.* Physical Review E, vol. 54, no. 4, pp. 3577–3580, 1996.

[RM92] M. Röckner and Z. M. Ma. *Introduction to the theory of (nonsymmetric) Dirichlet forms.* Springer, 1992.

[RS72] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York, 1972.

[RVE05] M. G. Reznikoff and E. Vanden-Eijnden. *Invariant Measures of SPDEs and Conditioned Diffusions.* Comptes Rendus Mathématique. Académie des Sciences. Paris, vol. 340, no. 4, pp. 305–308, 2005.

[SVW04] A. M. Stuart, J. Voss and P. Wiberg. *Conditional Path Sampling of SDEs and the Langevin MCMC Method*. Communications in Mathematical Sciences, vol. 2, no. 4, pp. 685–697, 2004.

[Tri83] H. Triebel. *Theory of Function Spaces*, vol. 78 of *Monographs in Mathematics*. Birkhäuser, Basel, 1983.

[Tri92] H. Triebel. *Theory of Function Spaces II*, vol. 84 of *Monographs in Mathematics*. Birkhäuser, Basel, 1992.

[Tri06] H. Triebel. *Theory of Function Spaces III*, vol. 100 of *Monographs in Mathematics*. Birkhäuser, Basel, 2006.

[Zab88] J. Zabczyk. *Symmetric Solutions of Semilinear Stochastic Equations*. In G. Da Prato and L. Tubaro (editors), *Stochastic Partial Differential Equations and Applications II*, vol. 1390 of *Lecture Notes in Mathematics*, pp. 237–256. Springer, 1988.